Chapter 3 Systems of Linear Differential Equations

$$\frac{dx}{dt} = ax + by \qquad a, b, c, d \in \mathbb{R}$$
$$\frac{dy}{dt} = cx + dy$$

- homogeneous . . . .
- with constant coefficients . . . .

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A \cdot \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

matrix multiplication

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{\underline{v}' = A \cdot \underline{v}}$$

→ What is special about these DEQs?

① $\underline{v}$ is a sol. of $\underline{v}' = A \cdot \underline{v}$ and $\lambda \in \mathbb{R}$ then $\lambda \underline{v}$ will solve the DEQ as well

$$(\lambda \underline{v})' = \lambda \cdot \underline{v}'$$
$$A \cdot \lambda \underline{v} = \lambda \cdot (A \cdot \underline{v})$$

② $\underline{v}, \underline{w}$ are sol to the DEQ, so is their sum $\underline{v} + \underline{w}$:

$$(\underline{v} + \underline{w})' = \underline{v}' + \underline{w}'$$

$$(\underline{v}+\underline{w})' = \underline{v}' + \underline{w}'$$

$$A\cdot(\underline{v}+\underline{w}) = A\cdot\underline{v} + A\cdot\underline{w}$$

From prior experience we expect the general solution to be of the form

$\underline{v} = c\cdot\underline{v}_1 + d\cdot\underline{v}_2$ $\quad c, d$ are constants

$\underline{v}_1, \underline{v}_2$ two solutions

Remember

$$y'' = -y$$

general solution: $y = c\sin t + d\cos t$

Program: find two "good" solutions $\underline{v}_1$ and $\underline{v}_2$ and then obtain the general solutions as

$$\underline{v} = \underbrace{c\underline{v}_1 + d\underline{v}_2}$$

linear combinations of $\underline{v}_1$ and $\underline{v}_2$

→ what does "good" mean?

"good" = one solutions is not a multiple of the other one.

f.i. $\sin t$, $\cos t$

A bad solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a 0 · multiple of any other solution

lin. algebra: solutions are linearly independent

How to find equilibrium points?

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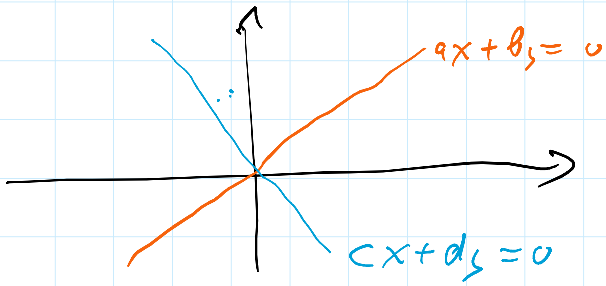
$$x' = ax + by$$
$$y' = cx + dy$$

equil. pts will satisfy

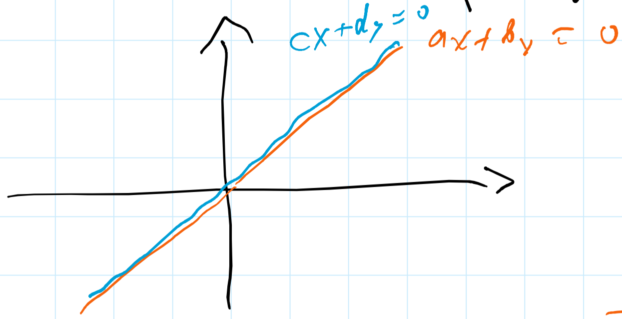
$$\left.\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}\right\}$$ system of two linear equations.

geometrically: intersection of two lines



both lines go through the origin

under normal circumstances $\binom{0}{0}$ will be the only solution (equil. pt.)



$y = -\frac{c}{d}x$

$y = -\frac{a}{b}x$

$\Rightarrow \quad \frac{a}{b} = \frac{c}{d}$

$b \neq 0, \ d \neq 0$

this is decided by the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Ex $\det \begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix} = 2 \cdot 1 - (3)(-4) = \underline{14}$

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very important

$\det A \neq 0$ means $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only equil. pt.

$\det A = 0$ means there are infinitely many equil pts; they form a line thru the origin.

For the time being, we will assume that

$\det A \neq 0$,

i.e. $\underline{v}' = A \cdot \underline{v}$ has only equil pt at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Looking at the direction fields of a few examples, they seem to have two pairs of straight line solutions

$\rightarrow$ This should correspond to our quest to find two good solutions

Q How do you find straight line solutions? (SLS)