HW §1. #34

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = x$$

not a homogeneous linear system

so the lin. principle does not necessarily apply!

$Y(t) = (t, \frac{t^2}{2})$ is a solution:

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = t = x$$ ✓ ✓

$Y(t) = (2t, t^2)$ is not a solution:

$$\frac{dx}{dt} = 2$$

this system is a non-homogeneous system of lin. DEQ

$$\frac{dx}{dt} = ax + by + f(t)$$
$$\frac{dy}{dt} = cx + dy + g(t)$$

$$\left|\begin{array}{l} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{array}\right.$$

$a, b, c, d \in \mathbb{R}$

$$\frac{dy}{dt} = cx + dy$$

$$Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$Y'(t) = A \cdot Y(t)$$

$\to$ Linearity Principle

If we know two "good" solutions
$Y_1(t)$ & $Y_2(t)$

① then the general solution will be

$$Y(t) = p \cdot Y_1(t) + q\, Y_2(t) \quad , p, q \in \mathbb{R}$$

②
$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

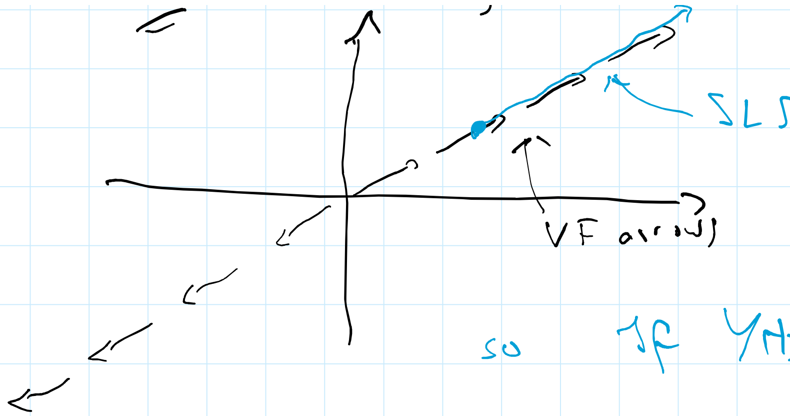
- has the unique solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
  if $\det A \neq 0$

$$\det A = ad - bc$$

- has non-zero solutions
  if $\det A = 0$

③ If we are lucky the system will have straight line solutions

$\to$ Q? How to find those SLS's ?




so If $Y(t)$ is a straight line solution

then $Y'(t) \parallel VF$

$$Y'(t) \parallel A \cdot Y(t)$$

$$A\,Y(t) = \lambda \cdot Y(t)$$

what about finding a vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ that does that

$$A\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$A \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \lambda\, Id \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda\, Id)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We will be able to find such a

We will be able to find such a $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ only if

$$\det(A - \lambda \, \mathrm{Id}) = 0$$

$$\det\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$\Longleftrightarrow \quad (a-\lambda)(d-\lambda) - bc = 0$$

$$\Longleftrightarrow \quad (\lambda - a)(\lambda - d) - bc = 0$$

$$\Longleftrightarrow \quad \lambda^2 - (a+d)\lambda + ad - bc = 0$$

this is a quadratic equation in $\lambda$

If we are lucky, we will get two solutions for $\lambda$

Example: $\quad A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ $\qquad \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\det(A - \lambda \, \mathrm{Id}) = 0$$

$$\det \begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} = 0$$

$$\Longleftrightarrow \quad (3-\lambda)(2-\lambda) - 2 = 0$$

$$\Longleftrightarrow \quad (\lambda-3)(\lambda-2) - 2 = 0$$

$$\Longleftrightarrow \quad \lambda^2 - 5\lambda + 6 - 2 = 0$$

$$\Longleftrightarrow \quad \lambda^2 - 5\lambda + 4 = 0$$

$$\Leftrightarrow \quad \lambda^2 - 5\lambda + 4 = 0$$

$$\Leftrightarrow \quad (\lambda - 4)(\lambda - 1) = 0$$

$$\Leftrightarrow \quad \lambda = 4 \quad \text{or } \lambda = 1$$

these two values are called the eigenvalues of the matrix $A$

If $\lambda$ is an eigenvalue then $A \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ will have non-zero solutions

$\underline{\underline{\lambda = 4}}$ $\quad \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 4 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$\begin{cases} 3x_0 + y_0 = 4x_0 \\ 2x_0 + 2y_0 = 4y_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -x_0 + y_0 = 0 \\ 2x_0 - 2y_0 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y_0 = x_0 \\ y_0 = x_0 \end{cases}$$

this is expected!

a good example for $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is called an eigenvector for eigenvalue $4 = \lambda$

for eigenvalue $4 = \lambda$

this is the direction of one pair of SLSs

$\underline{\lambda = 1}$

$$(A - \lambda \, Id)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_0 + y_0 = 0$$

$$y_0 = -2x_0$$

$$\left.\begin{array}{l} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ \text{or } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \end{array}\right\}$$ is an eigenvector for $\lambda = \underline{1}$

Can't take $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ this is a bad eigenvector (it is not considered an eigenvector at all)

what will a SLS look like?

$$Y(t) = f(t)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$ ← eigenvector for eigenvalue $\lambda$

$\quad f(t)\begin{pmatrix} \\ y_0\end{pmatrix}$ for eigenvalue $\lambda$

$$Y'(t) = f'(t)\begin{pmatrix}x_0\\ y_0\end{pmatrix}$$

$$\shortparallel$$

$$A\cdot Y(t) = A\cdot f(t)\cdot\begin{pmatrix}x_0\\ y_0\end{pmatrix}$$

$$= f(t)\cdot \underline{A\cdot\begin{pmatrix}x_0\\ y_0\end{pmatrix}}$$

$$= f(t)\ \overline{\lambda\begin{pmatrix}x_0\\ y_0\end{pmatrix}}$$

so $\boxed{f'(t)\begin{pmatrix}x_0\\ y_0\end{pmatrix} = f(t)\cdot\lambda\begin{pmatrix}x_0\\ y_0\end{pmatrix}}$ $\quad\begin{pmatrix}x_0\\ y_0\end{pmatrix}\neq\begin{pmatrix}0\\ 0\end{pmatrix}$

thus $f'(t) = \lambda f(t)$

one solution for $f$:

$$f(t) = e^{\lambda t}$$

Congrats! Our straight line solution will be $Y(t) = \begin{pmatrix}x_0\\ y_0\end{pmatrix}e^{\lambda t}$

$\lambda$ eigenv., $\begin{pmatrix}x_0\\ y_0\end{pmatrix}$ eigen-vector for $\lambda$

in our example

$$Y_1(t) = \begin{pmatrix}-1\\ 2\end{pmatrix}\cdot e^{t}$$
$$Y_2(t) = \begin{pmatrix}1\\ 1\end{pmatrix}e^{4t}$$

are 2 good straight line solutions

two good straight line solutions

the general solution of
$$Y' = \begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix} Y(t)$$

will be :
$$Y(t) = p\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{t} + q\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}$$
$$p, q \in \mathbb{R}$$

IV Problem  let's find the solution $Y(t)$ satisfying $x(0) = 2$, $y(0) = -3$

$$Y(0) = p\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{0} + q\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{0} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -p + q = 2 \\ 2p + q = -3 \end{cases}$$

$$\rightarrow \quad 3p = -5 \qquad q = 2 + \frac{5}{3} = \frac{11}{3}$$
$$p = -5/3$$

$$Y(t) = -\frac{5}{3}\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{t} + \frac{11}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}$$

or

$$\begin{cases} x(t) = \frac{5}{3}e^{t} + \frac{11}{3}e^{4t} \\ y(t) = \frac{-10}{3}e^{t} + \frac{11}{3}e^{4t} \end{cases}$$

since both eigenvalues are positive
all non-trivial solutions move away from the equil. pt.

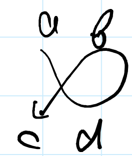
$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is called a SOURCE

$\binom{0}{0}$ is called a SOURCE

A SADDLE is next

$$Y'(t) = A \cdot Y(t) \qquad A = \begin{pmatrix} 3 & 3 \\ 4 & 3 \end{pmatrix}$$

eigenvalues $\quad \det(A - \lambda\, \mathrm{Id}) = 0$



$$\iff \det \begin{pmatrix} 3-\lambda & 3 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$\iff (3-\lambda)^2 - 12 = 0$$

$$\iff \lambda^2 - 6\lambda + 9 - 12 = 0$$

$$\iff \lambda^2 - 6\lambda - 3 = 0$$

characteristic equation

characteristic polynomial for A

Use the quadratic formula

$$\lambda = 3 \pm \sqrt{9+3} = 3 \pm \sqrt{12}$$
$$= 3 \pm 2\sqrt{3}$$

$\lambda = 3 + \sqrt{12}$

$$(A - \lambda\, \mathrm{Id}) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left( \begin{pmatrix} 3 & 3 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 3+\sqrt{12} & 0 \\ 0 & 3+\sqrt{12} \end{pmatrix} \right) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 - (3+\sqrt{12}) & 3 \\ 4 & 3 - (3+\sqrt{12}) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \cdots \\ 4 & -\sqrt{12} \end{pmatrix}\begin{pmatrix} \cdot \\ y_0 \end{pmatrix} = \begin{pmatrix} \cdot \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2\sqrt{3} & 3 \\ 4 & -2\sqrt{3} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left( (-2\sqrt{3}) \cdot x = 4 \qquad x = \frac{-2\cdot\sqrt{3}}{3} \qquad 3\left(\frac{-2\sqrt{3}}{3}\right) = -2\sqrt{3} \ \checkmark \right)$$

the two lines are multiples of each other

$$-2\sqrt{3}\, x_0 + 3 y_0 = 0$$

One good eigenvector: $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3 \\ +2\sqrt{3} \end{pmatrix}$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2\sqrt{3} \end{pmatrix}$$

One SLS: $\quad Y_1(t) = \begin{pmatrix} 3 \\ 2\sqrt{3} \end{pmatrix} e^{(3+\sqrt{12})t}$

$\underline{\lambda = 3 - \sqrt{12}}$ : $\quad A = \begin{pmatrix} 3 & 3 \\ 4 & 3 \end{pmatrix}$

eigenvector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ has to satisfy

$$(A - \lambda\, \mathrm{Id})\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-(3-\sqrt{12}) & 3 \\ 4 & 3-(3-\sqrt{12}) \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sqrt{12}\, x_0 + 3 y_0 = 0$$

$$\begin{pmatrix} x_0 \\ \cdot \end{pmatrix} = \begin{pmatrix} 3 \\ \cdot \end{pmatrix}$$

$\sqrt{12}\ x_0 + 3y_0 = 0$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3 \\ -\sqrt{12} \end{pmatrix}$$

2^nd SLS $\quad Y_2(t) = \begin{pmatrix} 3 \\ -\sqrt{12} \end{pmatrix} e^{(3-\sqrt{12})t}$

$$3+\sqrt{12} > 0$$
$$3-\sqrt{12} < 0$$