Test on Wednesday

Office hours M 3–4
T $4^{30} - 5^{30}$

3.2 #14

$$Y' = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} Y \qquad Y(0) = Y_0$$

$$Y(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{(B)}$$

→ Find λ (B), solving the characteristic equation:

$$\det(A - \lambda\, Id) = 0$$

$$\Leftrightarrow (4-\lambda)(1-\lambda) - (-2)\cdot 1 = 0 \qquad A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

$$\Leftrightarrow (\lambda-4)(\lambda-1) + 2 = 0$$

$$\Leftrightarrow \lambda^2 - 5\lambda + 4 + 2 = 0$$

$$\Leftrightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Leftrightarrow (\lambda-2)(\lambda-3) = 0$$

$$\Leftrightarrow \lambda = 2 \text{ or } \lambda = 3 \qquad \text{"eigenvalues"}$$

→ compute eigenvectors for each eigenvalue

$\underline{\lambda = 2}$ $\quad A - \lambda Id = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \quad \det\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} = 0\,!$

$$\begin{pmatrix} 2 & -2 \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} x \\ \cdot \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad x - y = 0 \Leftrightarrow x = y$$

we can take f.i. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = 3}$ $\quad A - \lambda \, \mathrm{Id} = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad x - 2y = 0 \Leftrightarrow x = 2y$$

we can take f.i. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\rightarrow$ general solution

$$Y(t) = C \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + D \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}$$

$\rightarrow$ Solve the initial value problem:

$Y(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \rightarrow \quad C = 0, \; D = 1$

$$Y(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}$$

$Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = Y(0) = C \begin{pmatrix} 1 \\ 1 \end{pmatrix} + D \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad C = -1 \quad D = 1$$

$$Y(t) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}$$

3.2 # 19

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$ 2nd order lin. DEQ $p, q > 0$

→ Convert into a system

$x = \frac{dy}{dt}$

$x' + px + qy = 0$

$\Leftrightarrow \; x' = -px - qy$
$\qquad y' = x$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -p & -q \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ $A = \begin{pmatrix} -p & -q \\ 1 & 0 \end{pmatrix}$

→ characteristic polynomial $A - \lambda Id$

$= \begin{pmatrix} -p-\lambda & -q \\ 1 & 0-\lambda \end{pmatrix}$ det

$(-p-\lambda)(-\lambda) + q = 0$

$\Leftrightarrow (\lambda + p)\cdot\lambda + q = 0$

$\Leftrightarrow \lambda^2 + p\lambda + q = 0$

$y'' + py' + qy = 0$

→ when do we get two distinct real eigenvalues?

Δ = discriminant > 0

$$\lambda = -\frac{p}{2} \pm \sqrt{\underbrace{\frac{p^2}{4} - q}_{\Delta}}$$

$\frac{p^2}{4} - q > 0$

$\Leftrightarrow \boxed{p^2 > 4q}$

$\iff \boxed{p^2 > 4q}$

$\rightarrow$ Check: eigenvalues are negative if they are real

eigenvalues real $\Rightarrow$ $\frac{p^2}{4} - q > 0$ i.e. $\frac{p^2}{4} - q < \frac{p^2}{4}$

$\iff \sqrt{\frac{p^2}{4} - q} < \frac{p}{2}$

$$\lambda = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

$p, q > 0$

$\lambda = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q} < 0$ ✓

$\lambda = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q} < -\frac{p}{2} + \frac{p}{2} = 0$ ✓

2.2 #14

$$\frac{dR}{dt} = 4R - 7F - 1$$

$$\frac{dF}{dt} = 3R + 6F - 12$$

find the equilibrium points of the system!

(1) $\frac{dR}{dt} = 0 \iff 4R - 7F = 1$

(2) $\frac{dF}{dt} = 0 \iff 3R + 6F = 12$

$3 \times (1)$ $12R - 21F = 3$

$4 \times (2)$ $12R + 24F = 48$

$(3) = (2) - (1)$ $45F = 45$

(1) $12R - 21F = 3$

(3) $F = 1$

$$(3) \quad F = 1$$

$$\Leftarrow (1) \quad 12R - 21 = 3$$

$$12R = 24$$

$$R = 2$$

1 equil pt: $R = 2, F = 1$ $\quad \begin{pmatrix} R \\ F \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2.2. #16

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x - x^3 - y$$

Again: Find the equil. pts!

$$\begin{cases} y = 0 \\ x - x^3 - y = 0 \end{cases}$$

Follow lazy person's rule

$y = 0$ ✓

$$\Leftarrow \quad x - x^3 = 0$$

$$\Leftrightarrow x(1 - x^2) = 0$$

$$\Leftrightarrow x(1-x)(1+x) = 0$$

3 solutions for x: $x = 0$ or $x = 1$ or $x = -1$

3 equil. pts. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Review the Linearity Principle

Review the Linearity Principle

$$Y' = A \cdot Y$$

linear (homogeneous) system

then 1) if $Y(t)$ is a solution
then $C \cdot Y(t)$ is also a solution
for all $C \in \mathbb{R}$

2) If $Y_1(t)$ and $Y_2(t)$ are solutions,
then $Y_1(t) + Y_2(t)$ is also a solution.

Existence and Uniqueness theorem

The solutions of "nice" autonomous system do not intersect in the phase plane (unless the phase portraits are identical)