6

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \quad \text{symmetric}$$

characteristic equation:

$$(\lambda - a)(\lambda - d) - b^2 = 0$$

$$\Leftrightarrow \quad \lambda^2 - (a+d)\lambda + (ad - b^2) = 0$$

$$\lambda = \frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2}{4} - (ad - b^2)}$$

$$= \frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2 - 4ad}{4} + b^2}$$

$$\boxed{(a+d)^2 - 4ad = a^2 + 2ad + d^2 - 4ad = a^2 - 2ad + d^2 = (a-d)^2}$$

$$\Downarrow$$

$$\lambda = \frac{a+d}{2} \pm \sqrt{\underbrace{\frac{(a-d)^2}{4} + b^2}_{\geq 0 \text{ for all } a, b, d\,!}}$$

P.S. If $b \neq 0$, then $b^2 > 0$ so

$$\frac{(a-d)^2}{4} + b^2 > 0$$

so we obtain 2 distinct real eigenvalues.

real eigenvalues.

#4 Find equilibrium points:

(1) $x' = 2x\left(1 - \frac{x}{2}\right) - xy$

(2) $y' = 3y\left(1 - \frac{y}{3}\right) - 2xy$

(1) $2x\left(1 - \frac{x}{2}\right) - xy = 0$

(2) $3y\left(1 - \frac{y}{3}\right) - 2xy = 0$

(1) $2x \cdot \left(1 - \frac{x}{2} - \frac{y}{2}\right) = 0$

$\underline{x = 0}$

$\overset{(2)}{\Longrightarrow}$ $3y\left(1 - \frac{y}{3}\right) = 0$

$\underline{y = 0}$ $\quad$ $\underline{y = 3}$

2 equil. pts

$(0,0)$, $(0,3)$

no animals at all.

y animal wipes out the x animal

$\underline{x + y = 2}$ $\iff x = 2 - y$

$\overset{(2)}{\Longrightarrow}$

$3y\left(1 - \frac{y}{3}\right) - 2(2 - y)y = 0$

$\iff 3y - y^2 - 4y + 2y^2 = 0$

$\iff y^2 - y = 0$

$\iff y(y - 1) = 0$

$y = 0$ $\Rightarrow x = 2$

$y = 1$, $x = 1$

2 more equil. pts:

$(?,0)$ and $(1,1)$

x animal wipes out the y animal

coexistence

#3

$$\begin{cases} x' = 2x & (1) \\ y' = x^2 - 3y & (2) \end{cases}$$

<u>not</u> a linear system of DEQ's (b/c of the x^2)

partially decoupled

(1) $x' = 2x$

$\Rightarrow x = Ae^{2t}$ $\quad$ A a const.

$\hookrightarrow (2)$ $y' = (A^2e^{4t}) - 3y$

not arbitrary!

lin. DEQ:

$$y' + 3y = A^2e^{4t} \quad \Big| \cdot e^{3t}$$

$$(ye^{3t})' = A^2e^{7t}$$

$$ye^{3t} = \frac{A^2}{7}e^{7t} + B$$

$$\Rightarrow y = \frac{A^2}{7}e^{4t} + Be^{-3t}$$

$$\left[\begin{array}{l} x = Ae^{2t} \\ y = \dfrac{A^2}{7}e^{4t} + Be^{-3t} \end{array}\right]$$

$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ $\quad Y' = A \cdot Y$ will turn out not to have SLS

what to do?

$\hookrightarrow$ Ignore the problem

find eigenvalues:

$$\lambda^2 - 1(-4) = 0$$

$$\Leftrightarrow \quad \lambda^2 + 4 = 0$$

$$\Leftrightarrow \quad \lambda = \pm 2i$$

$\hookrightarrow$ Q what is e^{2it}? ✓

<u>L. Euler</u>

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots$$

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots$$

$$e^{it} = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} \cdots$$

$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right)$$

$$+ i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$

$$= \cos t + i \sin t$$

Euler's Formula $\quad \boxed{e^{it} = \cos t + i \sin t}$

so the general solution to our system should be

eigenvectors ?

$$Y(t) = C\begin{pmatrix} \ \end{pmatrix}e^{2it} + D\begin{pmatrix} \ \end{pmatrix}e^{-2it}$$

$$= C\begin{pmatrix} \ \end{pmatrix}(\cos 2t + i\sin 2t)$$

$$+ D\begin{pmatrix} \ \end{pmatrix}(\cos(-2t) + i\sin(-2t))$$

$$= C\begin{pmatrix} \ \end{pmatrix}\cos 2t + i\sin 2t$$

$$+ D\begin{pmatrix} \ \end{pmatrix}(\cos 2t - i\sin(2t))$$

let's compute an eigenvector next,
say for $\lambda = 2i$:

$$(A - \lambda \, \mathrm{Id}) = \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix}$$

$$\left(\det\begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} = -4 - (1)(-4) = 0 \right)$$

eigenvector: $\begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\leftarrow$ two lines are multiples of each other

$$\Leftrightarrow \quad -2ix + y = 0$$

$\Rightarrow$ we can take f.i. $\begin{pmatrix} x \\ y \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 2i \end{pmatrix}}$ suitable eigenvector for $\lambda = 2i$

$(-2i)(-2i) \quad 1(-2i) \leftarrow$ first line $\times (-2i)$

$-4 \quad -2i \quad$ = second line

Euler's Observation :

If $Y(t) = Y_1(t) + i\,Y_2(t)$ is a complex solution
(Y_1, Y_2 real-valued)

then both $Y_1(t)$ and $Y_2(t)$ will be real valued solutions of our system.

That means we are done!

$$Y(t) = \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} = \begin{pmatrix} 1 \\ 2i \end{pmatrix} \left(\cos 2t + i \sin 2t \right)$$

$$= \begin{pmatrix} \cos 2t + i \sin 2t \\ -2 \sin 2t + i\, 2 \cos 2t \end{pmatrix}$$

$$Y(t) = \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix}$$

is a solution (complex-valued)

so $Y_1(t) = \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix}$

and $Y_2(t) = \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix}$

solve our linear system;

they are clearly no multiples of each other

so the general solution will be

$$\boxed{Y(t) = C \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix} + D \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix}}$$

C, D arbitrary constants

C, D arbitrary constants

All i's are gone: full success story!

<u>Another example</u>:

$$A = \begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}$$

(1) <u>Compute the eigenvalues</u>:

$$(\lambda+3)(\lambda+3) - (-2)2 = 0$$

$$(\lambda+3)^2 + 4 = 0$$

$$(\lambda+3)^2 = -4$$

$$(\lambda+3) = \pm 2i$$

$$\lambda = -3 \pm 2i$$

(2) Discard one of the eigenvalues and compute an eigenvector for the other one.

$$\lambda = -3 - 2i$$

$$A - \lambda \mathrm{Id} = \begin{pmatrix} -3-(-3-2i) & -2 \\ 2 & -3-(-3-2i) \end{pmatrix}$$

$$= \begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix}$$

lines are multiples of each other!

eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ will satisfy

$$2x + 2iy = 0 \quad (2^{nd}\text{ line})$$

$$x = -iy$$

good choice: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

eigenvector for $\lambda = -3-2i$

(3) Compute one solution; write with Euler's Formula

$$y(t) = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(-3-2i)t}$$

$$= \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-3t} e^{-2it}$$

$$= \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-3t} \left(\cos{-2t} + i \sin{-2t} \right)$$

$$y(t) = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-3t} \left(\cos 2t - i \sin 2t \right)$$

(4) Separate real and imaginary part:

$$y(t) = e^{-3t} \begin{pmatrix} \sin 2t & -i\cos 2t \\ \cos 2t & -i \sin 2t \end{pmatrix}$$

$$y(t) = e^{-3t} \left[\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} - i \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} \right]$$

(5) Then the general soln

⑤ Reap the benefits

$$y_1(t) = e^{-3t}\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

$$y_2(t) = e^{-3t}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$$

will be two "good" real solutions!

⑥ General solution.

$$y(t) = C e^{-3t}\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} + D e^{-3t}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$$

Bingo!