Last example for complex eigenvalues:

$$A = \begin{pmatrix} +1 & -2 \\ 3 & 3 \end{pmatrix}$$

① Find the eigenvalues:

$$(\lambda - 1)(\lambda - 3) + 6 = 0$$

$$\Leftrightarrow \lambda^2 - 4\lambda + 3 + 6 = 0$$
$$\Leftrightarrow \lambda^2 - 4\lambda + 9 = 0$$
$$\Leftrightarrow (\lambda^2 - 4\lambda + 4) + 5 = 0$$
$$\Leftrightarrow (\lambda - 2)^2 = -5$$
$$\Leftrightarrow \lambda - 2 = \pm\sqrt{5} \cdot i$$
$$\Leftrightarrow \lambda = 2 \pm \sqrt{5}\, i$$

② Next step: Find an eigenvector for one of the λ's, say $\lambda = 2 + \sqrt{5}i$

$$(A - \lambda\, \mathrm{Id}) = \begin{pmatrix} 1-(2+\sqrt{5}i) & -2 \\ 3 & 3-(2+\sqrt{5}i) \end{pmatrix}$$

$$= \begin{pmatrix} -1-\sqrt{5}i & -2 \\ 3 & 1-\sqrt{5}i \end{pmatrix}$$

eigenvector $\binom{x}{y}$ has to satisfy

$$(-1-\sqrt{5}i)x - 2y = 0$$
$$\& \quad 3x - (1-\sqrt{5}i)y = 0$$

two lines are multiples of each other

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$$3x - (1-\sqrt{5}i)y = 0$$

a suitable eigenvector is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-\sqrt{5}i \\ 3 \end{pmatrix}$$

(3) write down the corresponding complex valued solution

$$y(t) = \begin{pmatrix} 1-\sqrt{5}i \\ 3 \end{pmatrix} e^{(2+\sqrt{5}i)t}$$

$$= \begin{pmatrix} 1-\sqrt{5}i \\ 3 \end{pmatrix} e^{2t} e^{\sqrt{5}it}$$ Euler's Formula

$$= \begin{pmatrix} 1-\sqrt{5}i \\ 3 \end{pmatrix} e^{2t}(\cos\sqrt{5}t + i\sin\sqrt{5}t)$$

(4) Separate into real part and complex part

no i's ... all i's

$$= \begin{pmatrix} e^{2t}\cos\sqrt{5}t \\ 3e^{2t}\cos\sqrt{5}t \end{pmatrix}$$

$$+ i \begin{pmatrix} -\sqrt{5}e^{2t}\cos\sqrt{5}t \\ +\sqrt{5}e^{2t}\sin\sqrt{5}t \\ 3\sin\sqrt{5}t \end{pmatrix}$$

$$\Big[\Big(\cos\sqrt{5}t \qquad \Big) \cdot \Big(-\sqrt{5}\cos\sqrt{5}t + \sqrt{5}\sin\sqrt{5}t \Big]$$

$$= e^{2t}\left[\underbrace{\begin{pmatrix} \cos\sqrt{5}\,t \\ 3\cos\sqrt{5}\,t \end{pmatrix}} + i \underbrace{\begin{pmatrix} -\sqrt{5}\cos\sqrt{5}t + \sqrt{5}\sin\sqrt{5}t \\ 3\sin\sqrt{5}\,t \end{pmatrix}} \right]$$

(5) Euler's Observation

both $Y_1(t) = e^{2t}\begin{pmatrix} \cos\sqrt{5}\,t \\ 3\cos\sqrt{5}\,t \end{pmatrix}$

and $Y_2(t) = e^{2t}\begin{pmatrix} -\sqrt{5}\cos\sqrt{5}t + \sqrt{5}\sin\sqrt{5}t \\ 3\sin\sqrt{5}\,t \end{pmatrix}$

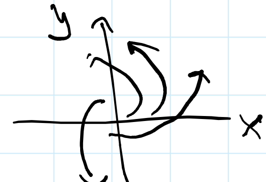
will be solutions to our DEQ system

(6) The general solution:

$Y(t) = C\,Y_1(t) + D\,Y_2(t)$ where C, D constants

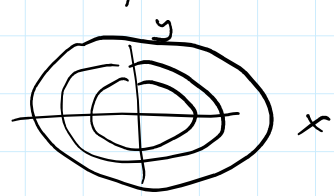
$\lambda = \alpha \pm \beta i$

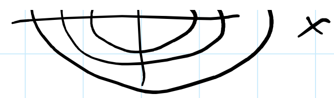
$\alpha > 0$ Spiral source



$\alpha < 0$ spiral sink

$\alpha = 0$ Center





Some Observations

1) $z = \alpha + i\beta$

then $\bar{z} = \alpha - i\beta$

z "conjugate"

→ If a real-valued polynomial has z as a root, then $\bar{z}$ will also be a root

Note $z = \bar{z} \iff z$ is a real number

2) if $\underline{v}$ is an eigenvector for the complex eigenvalue $\alpha + i\beta$

then $\overline{\underline{v}}$ is an eigenvector for $\alpha - i\beta$

$$\overline{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

0 as an eigenvalue:

Ex: $A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$

compute the eigenvalues:

compute the eigenvalues:

$$(\lambda-4)(\lambda-1)-4=0$$

$$\Leftrightarrow \quad \lambda^2-5\lambda+4-4=0$$

$$\Leftrightarrow \quad \lambda^2-5\lambda=0$$

$$\Leftrightarrow \quad \lambda(\lambda-5)=0$$

eigenvalues: $\lambda=0$, $\lambda=5$

compute eigenvectors for both

$\underline{\lambda=0}$

$$4x-2y=0$$

& $$-2x+y=0$$

$$\Leftrightarrow \quad 2x=y$$

suitable eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\underline{\lambda=5}$

$$A-5\,\mathrm{Id} = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}$$

$$A\underline{v}=\underline{0} \Leftrightarrow \quad -x-2y=0$$

$$\Leftrightarrow \quad -2y=x$$

a suitable choice:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

general solution to our DEQ system

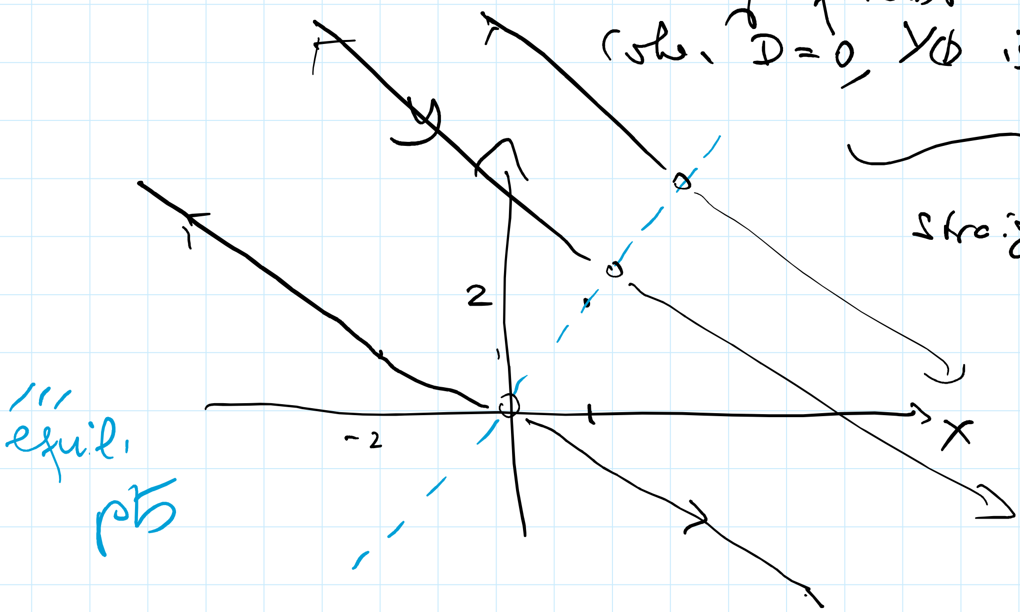
$$Y(t) = C\begin{pmatrix} 1 \\ \;\end{pmatrix} e^{0\cdot t} + D\begin{pmatrix} -2 \\ \;\end{pmatrix} e^{5t}$$

$$Y(t) = C\begin{pmatrix}1\\2\end{pmatrix}e^{0\cdot t} + D\begin{pmatrix}-2\\1\end{pmatrix}e^{5t}$$

$$= \underbrace{C\begin{pmatrix}1\\2\end{pmatrix}}_{\text{a line of equilibrium point}} + D\begin{pmatrix}-2\\1\end{pmatrix}e^{5t}$$

(when $D=0$, $Y(t)$ is constant)

straight line solution

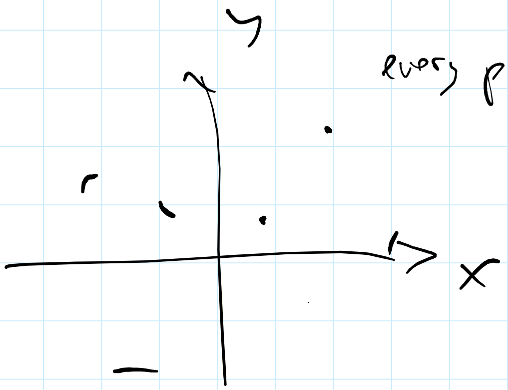


Ultimate 0-eigenvalue system

$$A = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

$\lambda = 0$ is an eigenvalue; it's repeated!

$$Y' = A\cdot Y = \begin{pmatrix}0\\0\end{pmatrix}$$



every point in the plane is an equil. pt.

Repeated eigenvalue:

Recipe

If $y' = A \cdot y$ has a repeated eigenvalue λ
with one line of straight line solutions
then the general solution will look as follows:

Let $\underline{v_0}$ be any vector
and set $\underline{v_1} = (A - \lambda Id)\underline{v_0}$

then

$$y(t) = e^{\lambda t}\underline{v_0} + te^{\lambda t}\underline{v_1}$$

will be the general solution!

nice feature: $\underline{v_0}$ are the initial conditions

Examples $y' = \begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix} \cdot y$

eigenvalues:
$$(\lambda+3)(\lambda+3) = 0$$

$\lambda = -3$ as repeated eigenvalue.

general solution: $y(t) = e^{-3t}\underline{v_0} + te^{-3t}\underline{v_1}$

$\underline{v_0} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $(A - \lambda Id)\begin{pmatrix} x \\ y \end{pmatrix} = \left(\overset{A}{\begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix}} - \overset{\lambda Id}{\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}} \right)$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{v_1} = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

$$\longrightarrow \quad y(t) = e^{-3t}\begin{pmatrix} x \\ y \end{pmatrix} + te^{-3t}\begin{pmatrix} 0 \\ x \end{pmatrix}$$

general solution.

let's check the solution for $\begin{matrix} x = 1 \\ y = 0 \end{matrix}$

$$y(t) = \begin{pmatrix} e^{-3t} \\ te^{-3t} \end{pmatrix}$$

$$y'(t) = \begin{pmatrix} -3e^{-3t} \\ e^{-3t} - 3te^{-3t} \end{pmatrix}$$

$$A\,y(t) = \begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} e^{-3t} \\ te^{-3t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{-3t} \\ e^{-3t} - 3(te^{-3t}) \end{pmatrix}$$

It's my lucky day!

Another example

$$y' = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$$

charact. equation:

$$(-2-\lambda)(-4-\lambda) + 1 = 0$$

$$\iff (\lambda+2)(\lambda+4) + 1 = 0$$

$$\iff \lambda^2 + 6\lambda + 9 = 0$$

$$\iff (\lambda+3)^2 = 0$$

$\Leftrightarrow \quad \lambda = -3$

$$y(t) = e^{\lambda t}\underline{v_0} + te^{\lambda t}\underline{v_1}$$

$$\underline{v_0} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \underline{v_1} = (A - \lambda \mathrm{Id})\underline{v_0}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x-y \\ x-y \end{pmatrix}$$

$$y(t) = e^{-3t}\begin{pmatrix} x \\ y \end{pmatrix} + te^{-3t}\begin{pmatrix} x-y \\ x-y \end{pmatrix}$$

"Easiest" example

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$\lambda = -2$ repeated eigenvalue

$A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$

then eigenvalues are a and d

eigenvectors have to satisfy

$$(A - \lambda \mathrm{Id})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

any vector in the plane is an eigenvector !

2nd order DEQ

$$y'' + py' + qy = 0$$

p, q are constants

convert into a system

$$y' = v$$
$$v' = y'' = -py' - qy$$
$$= -pv - qy$$

$$\begin{pmatrix} y \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \cdot \begin{pmatrix} y \\ v \end{pmatrix}$$

$$\left(\begin{array}{l} y' = 0 \cdot y + 1 \cdot v = v = y' \\ y'' = v' = -qy - py' \quad \checkmark \end{array} \right)$$

eigenvalues:

charact. eqn:
$$\lambda(\lambda + p) + q = 0$$
$$\Leftrightarrow \quad \lambda^2 + p\lambda + q = 0$$
$$\underline{y'' + py' + qy = 0}$$

Philosophy: you only about the y
not about the v,
since $v = y'$

Suppose λ_1, λ_2 are 2 distinct real eigenvalues
then

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = Y(t) = C \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda_1 t} + D \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} e^{\lambda_2 t}$$

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = Y(t) = C\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda_1 t} + D\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} e^{\lambda_2 t}$$

for suitable eigenvectors $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$$\boxed{y(t) = C^* e^{\lambda_1 t} + D^* e^{\lambda_2 t}}$$

general solution to $y'' + py' + qy = 0$ with eigenvalues λ_1, λ_2 distinct and real

Example: $y'' + 4y' + 3y = 0$

charact. eqn: $\lambda^2 + 4\lambda + 3 = 0$

$\iff (\lambda+1)(\lambda+3) = 0$

$\lambda = -1, \lambda = -3$

general solution:

$$y(t) = Ce^{-t} + De^{-3t}$$

let's check: $y'(t) = -Ce^{-t} - 3De^{-3t}$

$y''(t) = Ce^{-t} + 9De^{-3t}$

$$\begin{aligned} 4y'(t) + 3y(t) = &-4Ce^{-t} - 12De^{-3t} \\ &+ 3Ce^{-t} + 3De^{-3t} \\ \hline &-Ce^{-t} - 9De^{-3t} \end{aligned}$$

so $y'' + 4y' + 3y = 0$.

Case 1: 2 real eigenvalues distinct λ_1, λ_2

Sol.: $y(t) = Ce^{\lambda_1 t} + De^{\lambda_2 t}$

(2) 2 complex eigenvalues

$$\lambda = \alpha \pm i\beta$$

Sol: $y(t) = Ce^{\alpha t}\cos\beta t + De^{\alpha t}\sin\beta t$

$$y(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{(\alpha + i\beta)t}$$

$$= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

$$= \underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\alpha t}\cos\beta t} + i\underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\alpha t}\sin\beta t}$$

Example $\quad y'' - 2y' + 5y = 0$

ch. eqn $\quad \lambda^2 - 2\lambda + 5 = 0$

$\Leftrightarrow \quad (\lambda^2 - 2\lambda + 1) = -4$

$\quad (\lambda - 1)^2 = -4$

$\Leftrightarrow \quad \lambda - 1 = \pm 2i$

$\Leftrightarrow \quad \lambda = 1 \pm 2i$

general solution:

$$y(t) = C\cdot e^{t}\cos 2t + De^{t}\sin 2t$$