Q4 P1

$$\frac{dx}{dt} = -2x + \frac{1}{2}y \quad (1)$$

$$\frac{dy}{dt} = -y \quad (2)$$

$$(2) \quad \frac{dy}{dt} = -y \Rightarrow y = Ae^{-t}$$

$$(1) \quad \frac{dx}{dt} = -2x + \frac{A}{2}e^{-t}$$

$$x' + 2x = \frac{A}{2}e^{-t} \quad \Big| \cdot e^{2t}$$

$$(xe^{2t})' = \frac{A}{2}e^{t}$$

$$xe^{2t} = \frac{A}{2}e^{t} + B$$

$$x = \frac{A}{2}e^{-t} + Be^{-2t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}e^{-t} + B\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-2t}$$

eigenvalues : $-1, -2$ $\rightarrow$ $(0,0)$ will be a sink!

Sols: $A\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}e^{-t}$

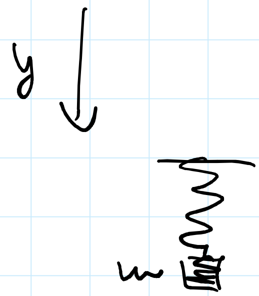
$B\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-2t}$

2^{nd} order DEQs

Most important examples:

spring mass systems

$$my'' + by' + ky = 0$$

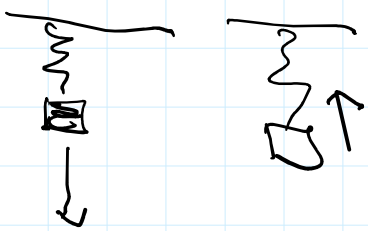


y position of the spring

m mass of the attachment to the spring

b damping factor (proportional to the velocity of the spring)

k spring constant (proportional to the stretch of the spring)



Hooke's Law

$$\underline{m>0,\ k>0,\ b\geq 0}$$

$$y'' + \frac{b}{m}y' + \frac{k}{m}y = 0$$

charact. equation: $\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$

$$\lambda = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

$$\lambda = -\frac{b}{2m} \pm \sqrt{\frac{b^2-4mk}{4m^2}}$$

Observations: – Both eigenvalues: $\lambda \leq 0$

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Clearly: $-\dfrac{b}{2m} - \sqrt{\cdots} < 0$

$$\frac{b^2}{4m^2} - \frac{k}{m} < \frac{b^2}{4m^2}$$

or $\dfrac{b}{2m} > \sqrt{\dfrac{b^2}{4m^2} - \dfrac{k}{m}}$

so the second eigenvalue ≤ 0 as well.

− if $b > 0$, both eigenvalues are negative

3 Cases

Overdamped Spring

1) $b^2 - 4mk > 0$

2 real eigenvalues, negative: λ_1, λ_2
solutions will be

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

Critically damped

weird case

2) $b^2 - 4mk = 0$

1 real eigenvalue repeated: λ

$$y(t) = Ae^{\lambda t} + Bte^{\lambda t}$$

normal case

Underdamped Spring

3) $b^2 - 4mk < 0$

complex eigenvalues: $\lambda = \alpha \pm i\beta$
($\alpha \le 0$)

$$y(t) = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t$$

$y(t) = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t$

4) $\alpha = 0 \iff \beta = 0$

undamped spring

$y(t) = A\cos\beta t + B\sin\beta t$

Trace - Determinant Plane

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad Y' = A \cdot Y$$

characteristic equation:

$$(\lambda - a)(\lambda - d) - bc = 0$$

$$\iff \lambda^2 - \underbrace{(a+d)}_{\text{trace } A}\lambda + \underbrace{(ad - bc)}_{\det A} = 0$$

$$\iff \lambda^2 - T\lambda + D = 0$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

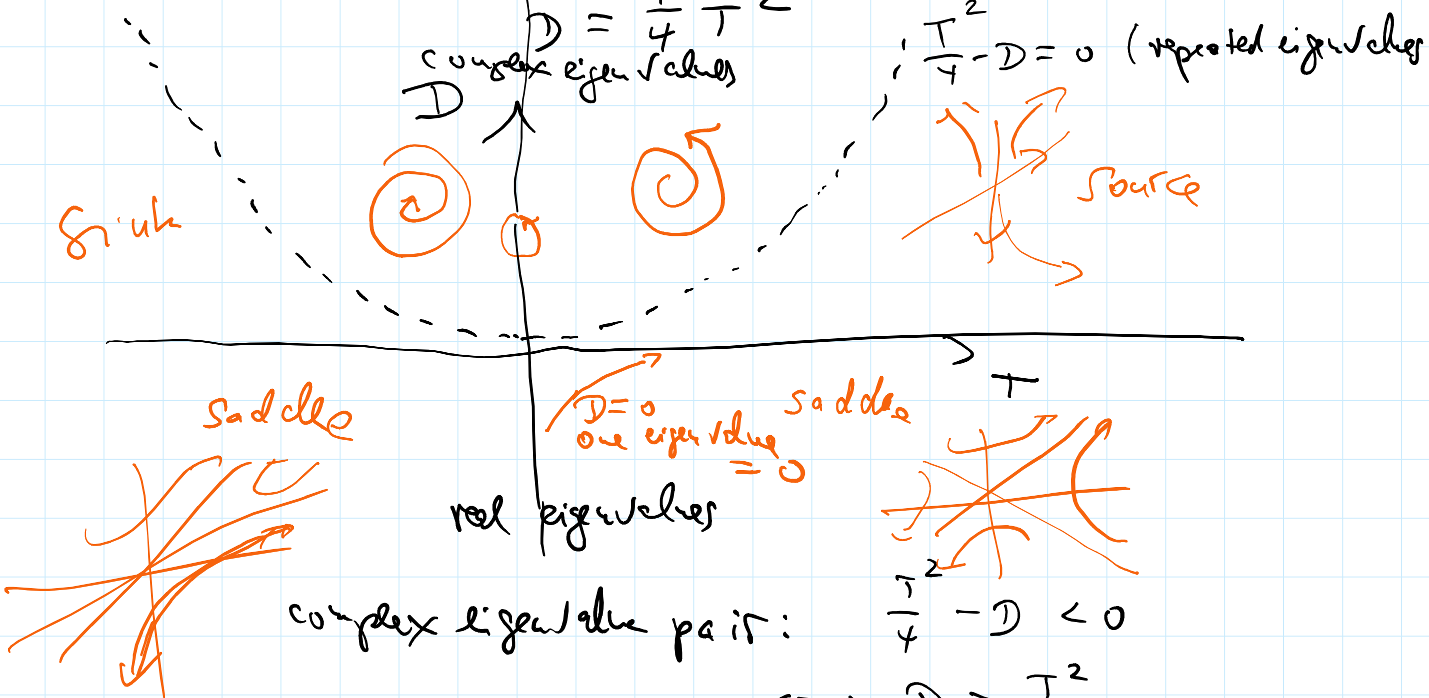
$D = \det A = ad - bc$
$T = \operatorname{tr} A = a + d$

T & D do not determine all coefficients of A, but they determine the eigenvalues

eigenvalues: $\lambda = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D}$

$\rightarrow$ when is $\frac{T^2}{4} - D = 0$?

$D = \frac{1}{4} T^2$



complex eigenvalue pair: $\frac{T^2}{4} - D < 0$

$\iff D > \frac{T^2}{4}$

$$\lambda = +\frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D}$$

when $T > 0$ at least one eigenvalue > 0

$T < 0$ at least one eigenvalue < 0

suppose $T > 0$ and $D > 0$

then the 2^{nd} eigenvalue > 0 as well

$$D > 0 \Rightarrow \frac{T^2}{4} - D < \frac{T^2}{4}$$

$$\sqrt{\frac{T^2}{4} - D} < \frac{T}{2}$$

so $\frac{T}{\ } - \sqrt{T^2 \ldots}$

so $\frac{T}{2} - \sqrt{\frac{T^2}{4} - D} > 0$