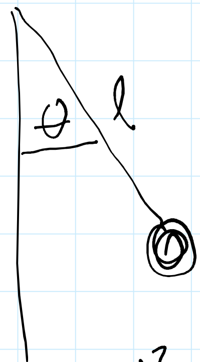
Non-linear System (Ch. V)

Examples Pendulum



θ angular position

θ' angular velocity

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{\ell}\sin\theta = 0$$

rewrite as a system

$$\theta' = v$$
$$v' (= \theta'') = -\frac{b}{m} v - \frac{g}{\ell} \sin\theta$$

non-linear

0 1
−1 +1

when θ is small
$$\sin\theta \approx \theta$$

Taylor series
$$\sin\theta = \theta - \underbrace{\frac{\theta^3}{3!} + \dots}_{\text{very small when } \theta \text{ is small}}$$

linearized pendulum:

$$\theta' = v$$
$$v' = -\frac{b}{m} v - \frac{g}{\ell}\theta$$

$$\begin{pmatrix}\theta \\ \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ g & \ell \end{pmatrix}\begin{pmatrix}\theta \\ \end{pmatrix}$$

$$\begin{pmatrix}\theta \\ v\end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{m}\end{pmatrix}\begin{pmatrix}\theta \\ v\end{pmatrix}$$

example $\begin{pmatrix}\theta \\ v\end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -1\end{pmatrix}\begin{pmatrix}\theta \\ v\end{pmatrix}$

what kind of equil. pt is $(0,0)$?

eigenvalues:

$$\lambda(\lambda+1) + 1 = 0$$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}$$

$\longrightarrow$ spiraling sink

! this will also be true for the pendulum itself!

what other equil. pts does the original pendulum have?

$$\theta' = v$$
$$v' = -v - \sin\theta$$

$\theta' = 0$ & $v' = 0$

$\theta' = 0 \Rightarrow v = 0$

$v' = 0 \Rightarrow \sin\theta \,\cancel{?}$

$\theta = 2\pi k$ $\quad$ $\theta = 2\pi k + \pi$

$(\theta = 0)$ $\quad$ $(\theta = \pi)$

So we linearize the pendulum at π

and ...

Can we linearize the pendulum at π as well

Yes

$$\sin\theta \approx \sin\pi + \sin'(\pi)\cdot(\theta-\pi) + \dots$$

$$= 0 + (-1)(\theta-\pi)$$

linearized at $\theta = \pi$

$$\begin{pmatrix}\theta\\ v\end{pmatrix}' = \begin{pmatrix}0 & 1\\ -(-1) & -1\end{pmatrix}\begin{pmatrix}\theta\\ v\end{pmatrix}$$

$$\begin{pmatrix}\theta\\ v\end{pmatrix}' = \begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\theta\\ v\end{pmatrix}$$

eigenvalues:

$$\lambda(\lambda+1) - 1 = 0$$

$$\Leftrightarrow \lambda^2 + \lambda - 1 = 0$$

$$\Leftrightarrow \lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+1}$$

$$\lambda_1 = -\frac{1}{2} + \sqrt{\frac{1}{4}+1} > 0$$

$$\lambda_2 = -\frac{1}{2} - \sqrt{\frac{1}{4}+1} < 0$$

saddle

The general method of linearization

$$x' = f(x,y)$$
$$y' = g(x,y)$$

autonomous non-linear system

$\downarrow$ linearized system at an equil. pt

↓ linearized system at an equil.pt (x_0, y_0) of the non-linear system

$$Y' = A \cdot Y$$

$$A = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix}$$

Jacobian matrix

the character of the equil.pt behavior of the linear system and the non-linear system are the same

(in normal cases)

Example

$$x' = -2x + y$$
$$y' = -y + x^2$$

$f(x,y) = -2x + y$

$g(x,y) = -y + x^2$

equil.pts:

$-2x + y = 0 \Rightarrow y = 2x$

$-y + x^2 = 0$

$-2x + x^2 = 0$

$\Leftrightarrow x(x-2) = 0$

$\Leftrightarrow x = 0$ or $x = 2$

$y = 0$ ~~or~~ $y = 4$

2 equil.pt $(0,0)$ and $(2,4)$

$$\text{Jac} = \begin{pmatrix} \frac{\partial f}{\partial x}(0,0) & \frac{\partial f}{\partial y}(0,0) \\ \frac{\partial g}{\partial x}(0,0) & \frac{\partial g}{\partial y}(0,0) \end{pmatrix}$$

linearize at $(0,0)$

$$\text{Jacobian} = \begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix} \text{ at } (0,0)\,!$$

Jacobian = $\left(\quad 2x \quad -1 \quad \right)$ at (0,0).

Jacobian at (0,0) = $\begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$

$$y' = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} \cdot y$$

triangular matrix

eigenvalues of linear System:

$\lambda = -2$ and $\lambda = -1$

(0,0) is a sink for lin. system
will behave like a sink
for the non-linear
system at (0,0).

at the equil.pt

Jac of (2,4): $\begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}$

$$y' = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}$$

eigenvalues

$$(\lambda+2)(\lambda+1) - 4 = 0$$

$$\lambda^2 + 3\lambda + 2 - 4 = 0$$

$$\lambda^2 + 3\lambda - 2 = 0$$

$$\lambda = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + 2}$$

saddle

non linear system close to
(2,4) ~~should~~ look like

$(2,4)$ ~~should~~ will look like a saddle

Example

$$x' = x(-x-3y+150) \quad (1)$$

$$y' = y(-2x-y+100) \quad (2)$$

Competing species!

$$\left(\begin{array}{l} x' = x(150-x) - 3xy \\ y' = y(100-y) - 2xy \end{array} \right)$$

equil. pts: (1) $x=0$ or $x+3y-150=0$

$x=0$ $\Rightarrow$ $y(100-y)=0$

$y=0$, $y=100$

2 equil. pts $(0,0)$, $(0,100)$

$x = 150-3y$ $\Rightarrow$ (2)

$$y(100-y) - 2(150-3y)y = 0$$

$$\Leftrightarrow 100y - y^2 - 300y + 6y^2 = 0$$

$$\Leftrightarrow 5y^2 - 200y = 0$$

$$\Leftrightarrow y=0 \text{ or } 5y=200$$

$$\Leftrightarrow y=0 \text{ or } y=40$$

$x=150$, $x=30$

2 more equil pts: $(150,0)$ and $(30,40)$

< more equil pts:

(150, 0) and (30, 40)

Jacobian matrix $f(x,y) = x' = -x^2 - 3xy + 150x$

$g(x,y) = y' = -2xy - y^2 + 100y$

$$Jac = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x - 3y + 150 & -3x \\ -2y & -2x - 2y + 100 \end{pmatrix}$$

at (0, 0) : $Jac = \begin{pmatrix} 150 & 0 \\ 0 & 100 \end{pmatrix}$

eigenvalues : 100, 150

source

at (0, 100) : $Jac = \begin{pmatrix} -150 & 0 \\ -200 & -100 \end{pmatrix}$

$(\lambda + 150)(\lambda + 100) - 0 = 0$

eigenvalues: $-150, -100$

Sink

at (150, 0) : $Jac = \begin{pmatrix} -150 & -450 \\ 0 & -200 \end{pmatrix}$

eigenvalues : $-150, -200$

at (30, 40) $Jac = \begin{pmatrix} -30 & -90 \\ -80 & -40 \end{pmatrix}$

$$(\lambda+30)(\lambda+40) - 80 \cdot 90 = 0$$

$$\lambda^2 + 70\lambda + 1200 - 7200 = 0$$

$$\lambda^2 + 70\lambda - 6000 = 0$$

$$\lambda = -35 \pm \sqrt{1225 + 6000}$$

saddle