Test 3 on Wednesday

Office Hours: today, after class
Tuesday 13:30 – 14:50
~~1630 – 1750~~

3.4 #6 $\quad Y' = \begin{pmatrix} 0 & 2 \\ -2 & -1 \end{pmatrix} \quad Y_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(1) Find eigenvalues:

$$\lambda(\lambda+1) + 4 = 0$$
$$\Leftrightarrow \quad \lambda^2 + \lambda + 4 = 0$$
$$\Leftrightarrow \quad \lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4}$$
$$\Leftrightarrow \quad \lambda = -\frac{1}{2} \pm \frac{i\sqrt{15}}{2}$$

complex eigenvalues!

(2) Compute an eigenvector for $\lambda = -\frac{1}{2} + i\frac{\sqrt{15}}{2}$

$$(A - \lambda \mathrm{Id}) = \begin{pmatrix} \frac{1}{2} - i\frac{\sqrt{15}}{2} & 2 \\ -2 & -\frac{1}{2} - i\frac{\sqrt{15}}{2} \end{pmatrix}$$

$$(A - \lambda \mathrm{Id})\, \underline{v_0} = \underline{0} \qquad v_0 = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left(\frac{1}{2} - i\frac{\sqrt{15}}{2}\right)x + 2y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{1}{2} + i\frac{\sqrt{15}}{2} \end{pmatrix}$$

(3) Write down a complex solution

(3) Write down a complex solution

$$Y(t) = \begin{pmatrix} 2 \\ -\frac{1}{2} + i\frac{\sqrt{15}}{2} \end{pmatrix} e^{\left(-\frac{1}{2} + i\frac{\sqrt{15}}{2}\right)t}$$

(4) Separate into real and imaginary parts

$$Y(t) = e^{-\frac{t}{2}} \begin{bmatrix} 2\left(\cos\frac{\sqrt{15}}{2}t + i\sin\frac{\sqrt{15}}{2}t\right) \\ \left(-\frac{1}{2} + i\frac{\sqrt{15}}{2}\right)\left(\cos\frac{\sqrt{15}}{2}t + i\sin\frac{\sqrt{15}}{2}t\right) \end{bmatrix}$$

$$Y(t) = e^{-\frac{t}{2}} \left[\begin{pmatrix} 2\cos\frac{\sqrt{15}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{15}}{2}t - \frac{\sqrt{15}}{2}\sin\frac{\sqrt{15}}{2}t \end{pmatrix} + i \begin{pmatrix} 2\sin\frac{\sqrt{15}}{2}t \\ -\frac{1}{2}\sin\frac{\sqrt{15}}{2}t + \frac{\sqrt{15}}{2}\cos\frac{\sqrt{15}}{2}t \end{pmatrix} \right]$$

! only one i in this spot!

(5) Using Euler's result

$$Y_1(t) = e^{-\frac{t}{2}} \begin{pmatrix} 2\cos\frac{\sqrt{15}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{15}}{2}t - \frac{\sqrt{15}}{2}\sin\frac{\sqrt{15}}{2}t \end{pmatrix}$$

and $$Y_2(t) = e^{-\frac{t}{2}} \begin{pmatrix} 2\sin\frac{\sqrt{15}}{2}t \\ -\frac{1}{2}\sin\frac{\sqrt{15}}{2}t + \frac{\sqrt{15}}{2}\cos\frac{\sqrt{15}}{2}t \end{pmatrix}$$

are both solutions to the DE system.

(6) The general solution:

$$Y(t) = C\,Y_1(t) + D\,Y_2(t)$$

(7) Solving the initial condition $Y_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$Y(0) = C\begin{pmatrix} 2 \\ -\frac{1}{2} \end{pmatrix} + D\begin{pmatrix} 0 \\ \frac{\sqrt{15}}{2} \end{pmatrix}$$

$2C = -1$

$-\frac{1}{2}C + \frac{\sqrt{15}}{2}D = 1$

5.1 #11 Equilibrium point analysis

$x' = x(-x-y+40)$ (1)

$y' = y(-x^2-y^2+2500)$ (2)

① Find the equilibrium points

(1) $x(-x-y+40)=0$

$\Leftrightarrow$

Case 1 $x=0$ or Case 2 $x+y=40$

$\hookrightarrow$ (2) $y(-y^2+2500)=0$

$y=0$ $y^2=2500$

$x=0\ y=0$ $\cancel{x=0,\ y=-50}$ $x=0,\ y=50$

not in Quadrant I

(1) $x=40-y$

(2) $y(-(y-40)^2-y^2+2500)=0$

$y=0$

$x=40,\ y=0$

$-(y-40)^2-y^2+2500=0$

$\Leftrightarrow -y^2+80y-1600-y^2+2500=0$

$\Leftrightarrow -2y^2+80y+900=0$

$$\Leftrightarrow \quad -2y^2 + 80y + 900 = 0$$

$$\Leftrightarrow \quad y^2 - 40y - 450 = 0$$

$$\Leftrightarrow \quad y = 20 \pm \sqrt{400 + 450}$$

$$\Leftrightarrow \quad y = 20 \pm \sqrt{850}$$

$$x = 20 + \sqrt{850} \qquad x = 20 - \sqrt{850}$$
$$y = 20 - \sqrt{850} \qquad y = 20 + \sqrt{850}$$

Compute the Jacobian

$$f(x, y) = -x^2 - xy + 40x$$
$$g(x, y) = -x^2 y - y^3 + 2500y$$

$$Jac = \begin{pmatrix} -2x - y + 40 & -x \\ -2xy & -x^2 - 3y^2 + 2500 \end{pmatrix}$$

at $(0,0)$ $\quad Jac = \begin{pmatrix} 40 & 0 \\ 0 & 2500 \end{pmatrix}$

eigenvalues : $40, 2500$ Source

at $(0, 50)$ $\quad Jac = \begin{pmatrix} -10 & 0 \\ 0 & -5000 \end{pmatrix}$

eigenv. : $-10, -5{,}000$ Sink

at $(40,0)$ Jac $= \begin{pmatrix} -40 & -40 \\ 0 & 900 \end{pmatrix}$

eigenv: $-40, 900$ saddle

+ 2 more equil. pts to check

3.3. #15 (3.2 #22)

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0 \iff y'' = -4y' - y$$

Sketch the phase portrait!

(1) Revert to a System!

$y' = v$

$\cancel{x'}\ \cancel{v'} = (y'') = -4\cancel{v}' - y$

$$\begin{pmatrix} y \\ \cancel{v}\ x \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} y \\ \cancel{v}\ x \end{pmatrix}$$

$\leftarrow$ very careful!

eigenvalues: $\lambda(\lambda+4)+1 = 0$

$\iff \lambda^2 + 4\lambda + 1 = 0$

$\iff \lambda = -2 \pm \sqrt{4-1}$

$\iff \lambda = -2 \pm \sqrt{3}$ sink!

eigenvectors:

$\lambda = -2+\sqrt{3} \approx -0.3$

$$\lambda = -2+\sqrt{3} \approx -0.3$$

$$(A-\lambda\,\mathrm{Id}) = \begin{pmatrix} 2-\sqrt{3} & 1 \\ -1 & -2-\sqrt{3} \end{pmatrix}$$

$$\begin{pmatrix} 2-\sqrt{3} & 1 \\ -1 & -2-\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad (2-\sqrt{3})\,x + y = 0$$

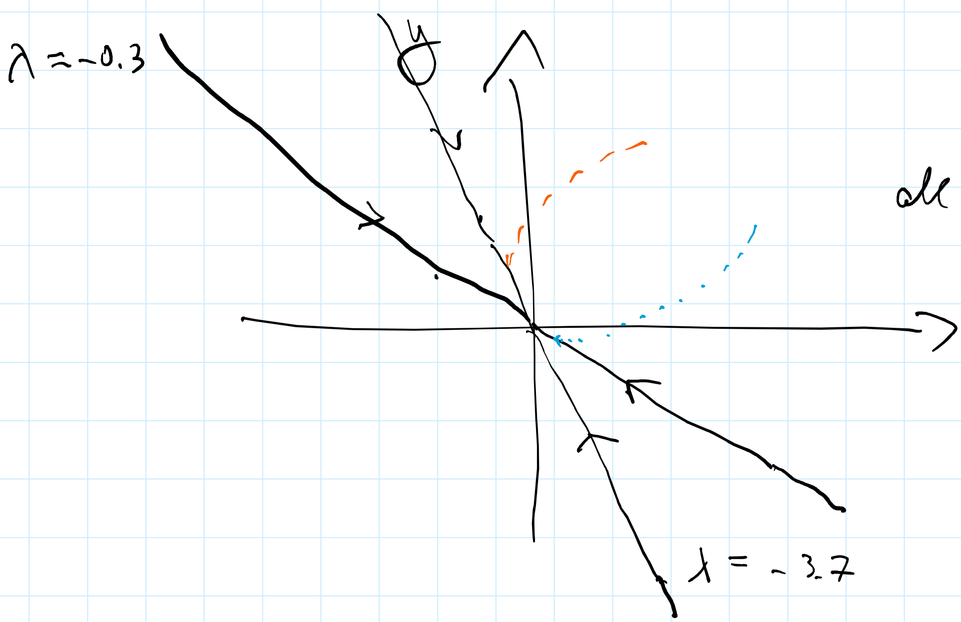
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2-\sqrt{3} \end{pmatrix} \approx \begin{pmatrix} -1 \\ 0.3 \end{pmatrix}$$

$$\lambda = -2-\sqrt{3} \approx -3.7$$

$$\begin{pmatrix} 2+\sqrt{3} & 1 \\ -1 & -2+\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad -x + (-2+\sqrt{3})\,y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2+\sqrt{3} \\ 1 \end{pmatrix} \approx \begin{pmatrix} -0.3 \\ 1 \end{pmatrix}$$



all solutions approach the equil. pt.

3.6 #32 $\quad \lambda$ is an eigenvalue for

3.6 #32

λ is an eigenvalue for

$$y'' + py' + qy = 0$$

Show that $V = \cancel{\begin{pmatrix} 1 \\ \lambda \end{pmatrix}}$ $\begin{pmatrix} \lambda \\ 1 \end{pmatrix}$ is an eigenvector

Convert to system:

$$y' = x$$

$$x' = y'' = -px - qy$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -p & -q \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues:

$$(\lambda + p)\lambda + q = 0$$

$$\cancel{\lambda^2 + p\lambda + q = 0}$$

$$\cancel{\lambda = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}}$$

$\begin{pmatrix} \lambda \\ 1 \end{pmatrix}$ $\cancel{\begin{pmatrix} 1 \\ \lambda \end{pmatrix}}$ is an eigenvector means

$$(A - \lambda\,\mathrm{Id})\cancel{\begin{pmatrix} 1 \\ \lambda \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -p-\lambda & -q \\ 1 & -\lambda \end{pmatrix} \cancel{\begin{pmatrix} 1 \\ \lambda \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -p-\lambda & -q \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} -p\lambda - \lambda^2 - q \\ 0 \end{pmatrix}$$

$\checkmark$

$$= \begin{pmatrix} -(\lambda^2 + p\lambda + q) \\ \end{pmatrix} = \begin{pmatrix} 0 \\ \end{pmatrix}$$

$$(A - \lambda \mathrm{Id}) \underline{v} = \begin{pmatrix} -(\lambda^2 + p\lambda + q) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

remember eigenvectors satisfy
$$(A - \lambda \mathrm{Id}) \underline{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3.7 #10

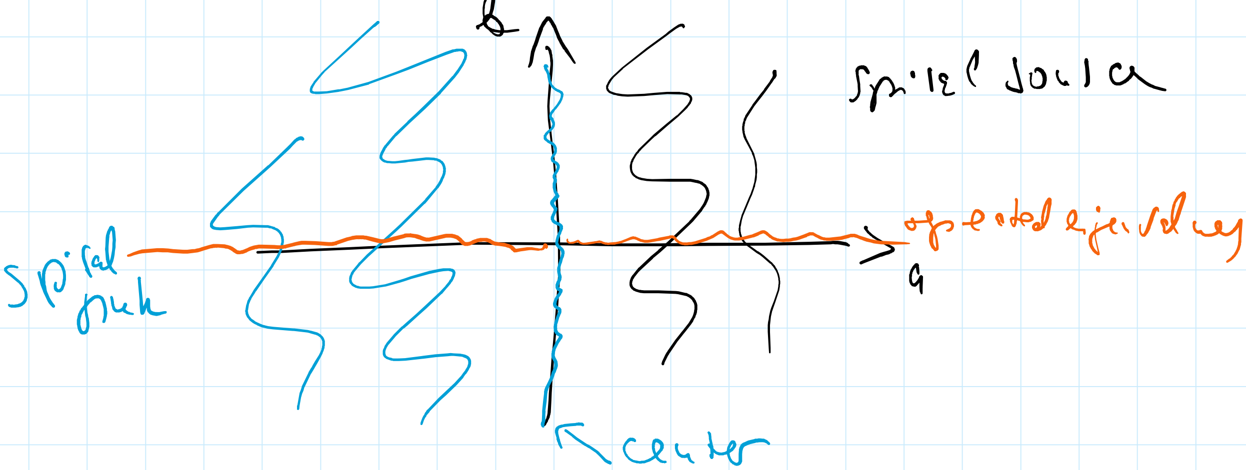
$$y' = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} y$$

characteristic equation:

$$(\lambda - a)^2 + b^2 = 0$$
$$\iff (\lambda - a)^2 = -b^2$$
$$\iff \lambda - a = \pm i b$$
$$\iff a \pm i b$$

$\underline{b \neq 0}$ If $a > 0$ spiral source
If $a < 0$ spiral sink
If $a = 0$ center

$\underline{b = 0}$ repeated eigenvalues



5.1 #6

competing species:

$$x' = 2x\left(1-\frac{x}{2}\right) - xy$$
$$y' = 3y\left(1-\frac{y}{3}\right) - 2xy$$

was done in class

3.4 #18

A real entries

complex: $\lambda = \alpha + i\beta$

$\overline{\lambda} = \alpha - i\beta \qquad \beta \neq 0$

show that eigenvectors must be complex

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \lambda = \alpha + i\beta$$

eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$: $\begin{pmatrix} a-(\alpha+i\beta) & b \\ c & d-(\alpha+i\beta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\underbrace{c}_{\text{real}}\, x + (\underbrace{d-\alpha}_{\text{real}} - i\beta)\, y = 0$$

$$cx = fy + i\beta y$$

no one of x or y has to be complex.

3.5 #18

$$Y' = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} Y$$

$$\left(\det\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} = 0 \iff \lambda = 0 \text{ is an eigenvalue} \right)$$

eigenvalues: $(\lambda-2)(\lambda-6) - 12 = 0$

$\iff \lambda^2 - 8\lambda + 12 - 12 = 0$

$$\Leftrightarrow \quad \lambda^2 - 8\lambda + 12 - 12 = 0$$

$$\Leftrightarrow \quad \lambda = 0 \text{ or } \lambda = 8$$

eigenvectors:

$\underline{\lambda = 0}$ $\quad \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$2x + 4y = 0 \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\Leftrightarrow \quad x + 2y = 0$$

$\underline{\lambda = 8}$ $\quad \begin{pmatrix} -6 & 4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$3x - 2y = 0 \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

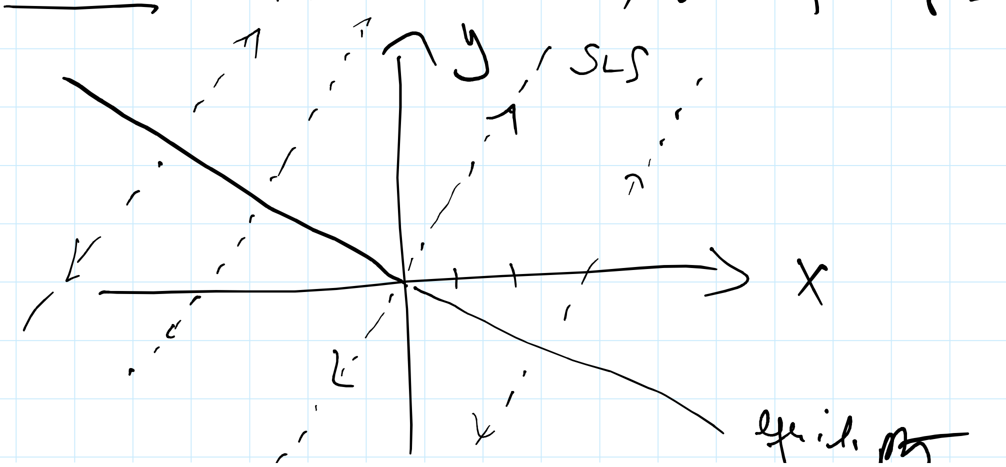
$$3x = 2y$$

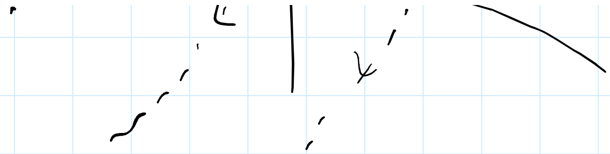
general solution:

$$y(t) = C\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{0 \cdot t} + D\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{8t}$$

$$= \underbrace{C\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\text{line of equil pts}} + \underbrace{D\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{8t}}_{\text{straight line sol.}}$$

$\underline{\text{If } D = 0}$: $\quad y(t) = C\begin{pmatrix} 2 \\ -1 \end{pmatrix}$; get equil. pts





Equil. pt