Another example of a nullcline-analysis

$$x' = 4 - x^2 - y^2$$
$$y' = 4 - x - 2y$$

x-nullcline

? $x' = 0$?

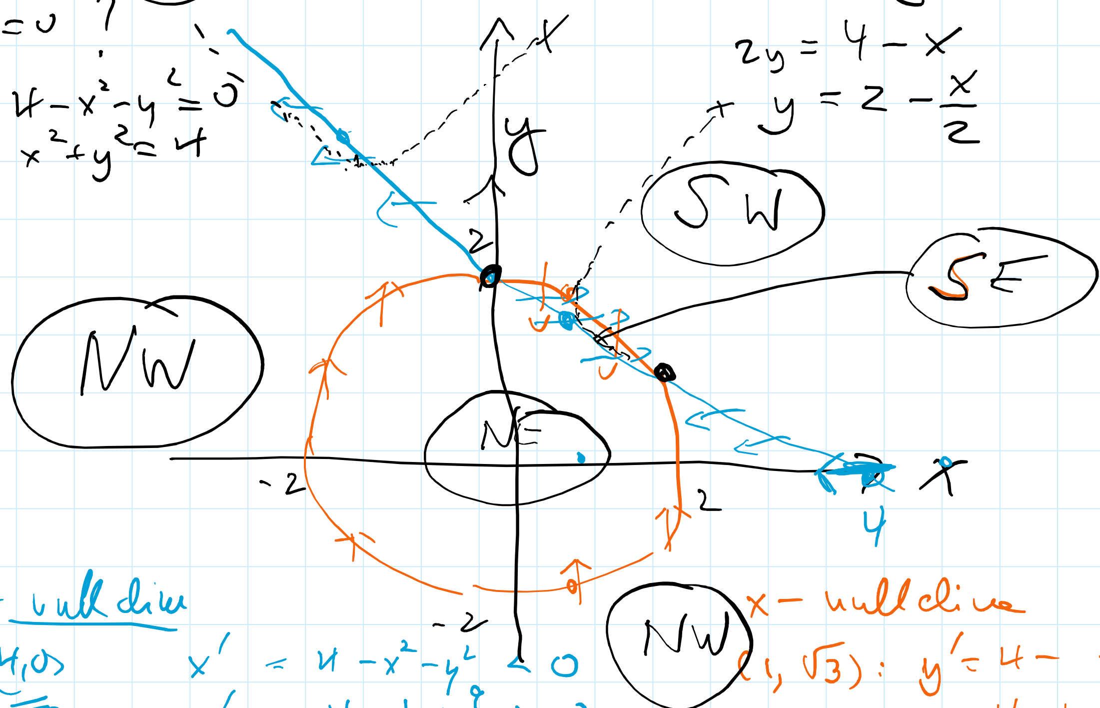
$4 - x^2 - y^2 = 0$

$\Leftrightarrow x^2 + y^2 = 4$

$y' = 0$ y-nullcline

$2y = 4 - x$

$y = 2 - \frac{x}{2}$



y-nullcline

$(4,0)$ $x' = 4 - x^2 - y^2 < 0$

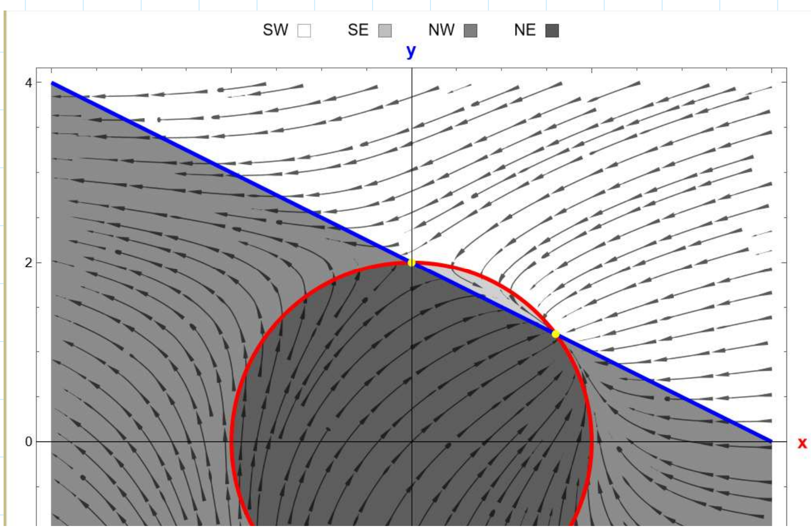
$(1, \frac{3}{2})$ $x' = 4 - 1 - \frac{9}{4} > 0$

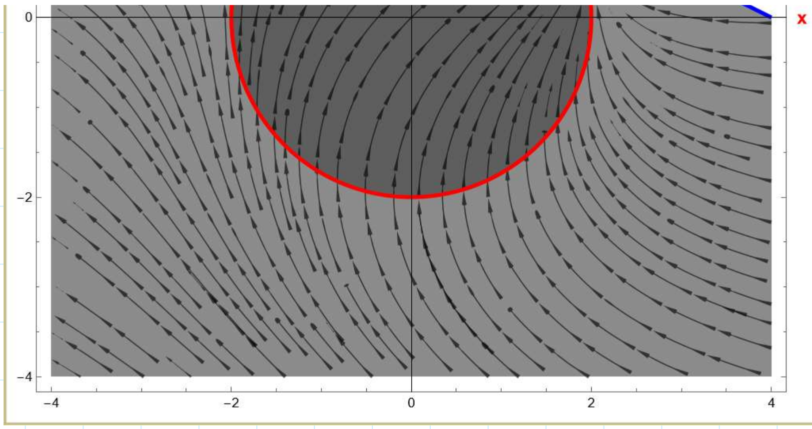
$(-2, 3)$ $x' = 4 - 4 - 9 < 0$

x-nullcline

$(1, \sqrt{3})$: $y' = 4 - x - 2y$
$= 4 - 1 - 2\sqrt{3}$
$\approx 4 - 1 - 3.4 < 0$

$(1, -\sqrt{3})$ $y' = 4 - x - 2y$
$= 4 - 1 + 2\sqrt{3} > 0$





The Laplace Transform

→ solve IVP only

→ Invented by Euler

Given a function $f(t)$ we compute a new function $Y(s)$, the Laplace transform

Definition: $$\boxed{Y(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$

Example: $f(t) = 1$

$$Y(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= \left(-\frac{1}{s} e^{-st}\right)\Bigg|_{t=0}^{t=\infty}$$

Don't worry

$$\Big)\Big|_{t=0}$$

$$= \lim_{t\to\infty}\left(-\frac{1}{s}e^{-st}\right) - \left(-\frac{1}{s}\right)$$

$$= 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

$y(t)$	$Y(s)$
1	$\frac{1}{s}$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$
t	$\frac{1}{s^2}$

Ex2 $\quad y(t) = e^{+\alpha t}$

$$Y(s) = \int_0^\infty e^{+\alpha t} e^{-st}\,dt$$

$$= \int_0^\infty e^{(\alpha - s)t}\,dt$$

$$= \left(\frac{1}{\alpha - s} e^{(\alpha - s)t}\right)\Big|_{t=0}^{t=\infty}$$

$$= 0 - \frac{1}{\alpha - s} = \frac{1}{s-\alpha}$$

Ex3 $\quad y(t) = t$

$$Y(s) = \int_0^\infty t e^{-st}\,dt$$

IbP $\quad u = t \quad dv = e^{-st}\,dt$

$\quad du = dt \quad v = -\frac{1}{s}e^{-st}$

$$= -\frac{1}{s}te^{-st}\Big|_{t=0}^{t=\infty} - \int_0^\infty \left(-\frac{1}{s}\right)e^{-st}\,dt$$

$$= (0 - 0) + \frac{1}{s}\left(-\frac{1}{s}e^{-st}\right)\Big|_{t=0}^{t=\infty}$$

$$= \frac{1}{s}\left(0 - \left(-\frac{1}{s}\cdot 1\right)\right)$$

$$= \frac{1}{s^2}$$

s^2

Instead of $Y(s)$ people write also

$$Y(s) = \mathcal{L}(f(t))(s) \qquad \mathcal{L}(t)(s) = \frac{1}{s^2}$$

$$\mathcal{L}(e^{\alpha t})(s) = \frac{1}{s-\alpha}$$

$\mathcal{L}$ is called the Laplace operator

input: $f(t)$

output $\int_0^\infty f(t)e^{-st}\,dt$

$\mathcal{L}$ is <u>linear</u>

$$\mathcal{L}(f(t)+g(t))(s)$$
$$= \mathcal{L}(f(t))(s) + \mathcal{L}(g(t))(s)$$

$$\mathcal{L}(c \cdot f(t))(s) = c \cdot \mathcal{L}(f(t))(s)$$

<u>Ex</u> $\mathcal{L}(5+3t)(s)$

$$= 5\mathcal{L}(1)(s) + 3\mathcal{L}(t)(s)$$
$$= 5 \cdot \frac{1}{s} + 3\frac{1}{s^2}$$
$$= \frac{5}{s} + \frac{3}{s^2}$$

$$\mathcal{L}(2e^{-2t} + 3e^{4t})(s)$$

$$= 2\mathcal{L}(e^{-2t})(s) + 3\mathcal{L}(e^{4t})(s)$$
$$= 2 \cdot \frac{1}{s+2} + 3\frac{1}{s-4}$$

$$= 2 \cdot \frac{1}{s+2} + 3 \frac{1}{s-4}$$

$$= \frac{2}{s+2} + \frac{3}{s-4}$$

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given a fct. $y(t)$

what is the Laplace transform of $y'(t)$?

$$\mathcal{L}(y'(t))(s)$$

$$= \int_0^\infty y'(t) e^{-st}\, dt$$

we'll use I.b.P

$u = e^{-st}$ $\quad dv = y'(t)\, dt$

$du = -se^{-st}\, dt$ $\quad v = y(t)$

$$= \left(y(t) e^{-st} \right) \Big|_{t=0}^{t=\infty}$$

$$+ s \int_0^\infty y(t) e^{-st}\, dt$$

$$= \left(0 - y(0) \right) + s \cdot \mathcal{L}(y(t))(s)$$

$$\boxed{\mathcal{L}(y'(t))(s) = s \cdot \mathcal{L}(y(t))(s) - y(0)}$$

Example $\quad \begin{array}{|l} y' = 4y \\ y(0) = 2 \end{array}$

$y(0) = 2$

$y' = 4y$ apply $\mathcal{L}$ on both sides

$\mathcal{L}(y'(t))(s) = \mathcal{L}(4y(t))(s)$

$s\mathcal{L}(y(t))(s) - y(0) = 4\mathcal{L}(y(t))(s)$

so $(s-4)\mathcal{L}(y(t))(s) = 2$

$\mathcal{L}(y(t))(s) = \frac{2}{s-4}$

table: $\mathcal{L}(e^{4t})(s) = \frac{1}{s-4}$

so $\mathcal{L}(2e^{4t})(s) = \frac{2}{s-4}$

$\mathcal{L}(y(t))(s) = \mathcal{L}(2e^{4t})(s)$

$\Rightarrow y(t) = 2e^{4t}$

$\mathcal{L}(y''(t))(s) = \mathcal{L}((y')'(t))(s)$

$= s\mathcal{L}(y'(t))(s) - y'(0)$

$= s(s\mathcal{L}(y(t))(s) - y(0)) - y'(0)$

$= s^2\mathcal{L}(y(t)) - s\cdot y(0) - y'(0)$

$$\boxed{\mathcal{L}(y'') = s^2\mathcal{L}(y) - s\cdot y(0) - y'(0)}$$