Laplace Transform

$$f(t) \xrightarrow{\mathcal{L}} \underbrace{\int_0^\infty f(t) e^{-st} \, dt}_{\text{function of } s}$$

limit at ∞ : $\underline{0}$?

Examples : $\mathcal{L}(e^{at})(s) = \frac{1}{s-a}$

$$\mathcal{L}(t)(s) = \frac{1}{s^2}$$

$$\mathcal{L}(f'(t))(s) = s\mathcal{L}(f(t))(s) - f(0)$$

more examples :

$f(t) = \sin(t)$, $\cos t$?

$$e^{i\beta t} = \cos\beta t + i\sin\beta t$$
$$e^{-i\beta t} = \cos\beta t - i\sin\beta t$$

$(+)$ $$\frac{e^{i\beta t} + e^{-i\beta t}}{2} = \cos\beta t$$

$(-)$ $$\frac{e^{i\beta t} - e^{-i\beta t}}{2i} = \sin\beta t$$

$$\mathcal{L}(\cos\beta t)(s) = \frac{1}{2}\mathcal{L}(e^{i\beta t})(s) + \frac{1}{2}\mathcal{L}(e^{-i\beta t})$$

$$= \frac{1}{2}\left(\frac{1}{s-i\beta} + \frac{1}{s+i\beta}\right)$$

$$= \frac{1}{2} \frac{(s+i\beta)+(s-i\beta)}{(s-i\beta)(s+i\beta)}$$

$$= \frac{1}{2} \frac{(s+i\beta)+(s-i\beta)}{(s-i\beta)(s+i\beta)}$$

$$= \frac{s}{s^2+\beta^2}$$

$$\mathcal{L}(\sin\beta t)(s) = \frac{1}{2i}\left(\mathcal{L}(e^{i\beta t})(s) - \mathcal{L}(e^{-i\beta t})(s) \right)$$

$$= \frac{1}{2i}\left(\frac{1}{s-i\beta} - \frac{1}{s+i\beta} \right)$$

$$= \frac{1}{2i} \frac{(s+i\beta)-(s-i\beta)}{(s-i\beta)(s+i\beta)}$$

$$= \frac{1}{2i} \frac{2i\beta}{s^2+\beta^2}$$

$$= \frac{\beta}{s^2+\beta^2}$$

Examples: $y'' + 2y' - 3y = 0$

$y(0) = 1$
$y'(0) = -1$

(#1)

$$\mathcal{L}(y'') = s\,\mathcal{L}(y') - y'(0)$$
$$= s(s\,\mathcal{L}(y) - y(0)) - y'(0)$$
$$= s^2\mathcal{L}(y) - s \cdot y(0) - y'(0)$$

$$\mathcal{L}(y') = s\,\mathcal{L}(y) - y(0)$$

apply Laplace transform

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + 2(s\mathcal{L}(y) - y(0))$$
$$- s \cdot 1 - (-1)$$
$$- 3\,\mathcal{L}(y) = 0$$

characteristic polynomial

$$\Leftrightarrow (s^2 + 2s - 3)\,\mathcal{L}(y) = s - 1 - 2$$

char. polynomial

$$\Longleftrightarrow (s^2+2s-3)\,\mathcal{L}(y) = s-1-2$$

$$\mathcal{L}(y) = \frac{s-3}{s^2+2s-3}$$

$$= \underbrace{\frac{s-3}{(s-1)(s+3)}}$$

task: recognize as Lapl. transforms!

use partial fractions:

$$\mathcal{L}(y)(s) = \frac{A}{s-1} + \frac{B}{s+3}$$

$$(\#2) \qquad \mathcal{L}(y)(s) = A\mathcal{L}(e^t) + B\mathcal{L}(e^{-3t})$$

$$= \mathcal{L}(Ae^t + Be^{-3t})$$

so $y(t) = Ae^t + Be^{-3t}$

Computing A & B:

$$\frac{s-3}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}$$

$$\frac{s-3}{(s-1)(s+3)} = \frac{A(s+3)+B(s-1)}{(s-1)(s+3)}$$

$$\Rightarrow s-3 = (A+B)s + (3A-B)$$

compare coefficients:

$$1 = A+B$$

$$3A - B = -3$$

$$B = 1-A$$

$$B = 1 - A$$

$$C_D = 3A - (1-A) = -3$$

$$4A = -2$$

$$A = -\frac{1}{2}, \quad B = \frac{3}{2}$$

our solution:

$$y(t) = -\frac{1}{2}e^{t} + \frac{3}{2}e^{-3t}$$

Example: $y'' + 2y' + 3y = e^{4t}$ $\quad y(0) = 0$, $y'(0) = 2$

apply $\mathcal{L}$ on both sides

$$(s^2 + 2s + 3)\mathcal{L}(y) - 2 = \frac{1}{s-4}$$

$$\mathcal{L}(y) = \frac{1}{(s-4)(s^2+2s+3)} + \frac{2}{s^2+2s+3}$$

$$= \frac{1}{(s-4)(s-1)(s+3)} + \frac{2}{(s-1)(s+3)}$$

part. fractions:

$$= \left(\frac{A}{s-4} + \frac{B}{s-1} + \frac{C}{s+3}\right) + \left(\frac{D}{s-1} + \frac{E}{s+3}\right)$$

$$y(t) = \underbrace{Ae^{4t}}_{\text{right hand side}} + \underbrace{Be^{t} + Ce^{-3t} + De^{t} + Ee^{-3t}}_{\text{are sol. to } y''+2y'+3y=0}$$

Example: $y'' + 2y' + 3y = e^{t}$ $\quad y(0) = 0$, $y'(0) = 2$

$$(s^2 + 2s + 3)\mathcal{L}(y) - 2 = \frac{1}{s-1}$$

$$\mathcal{L}(y) = \frac{2}{(s-1)(s+3)} + \frac{1}{(s-1)(s-1)(s+3)}$$

$$= \frac{2}{\quad} + \frac{1}{\quad}$$

$$= \frac{2}{(s-1)(s+3)} + \frac{1}{(s+3)(s-1)^2}.$$

(#9) $\mathcal{L}(t^n e^{at}) = \dfrac{n!}{(s-a)^{n+1}}$

$\mathcal{L}(te^t) = \dfrac{1}{(s-1)^2}$

$$\frac{2}{(s-1)(s+3)} + \frac{1}{(s+3)(s-1)^2}$$

$$= \left(\frac{A}{s-1} + \frac{B}{s+3}\right) + \left(\frac{C}{s+3} + \frac{D}{s-1} + \frac{E}{(s-1)^2}\right)$$

$$y(t) = Ae^t + Be^{-3t} + Ce^{-3t} + De^t + E \cdot te^t$$

Example: $y'' + y = 0$ $\quad y(0) = 1$, $y'(0) = 0$

$$(s^2+1)\mathcal{L}(y) - s \cdot \underset{y(0)}{1} - \underset{y'(0)}{0} - \underset{y(0)}{1} = 0$$

$$(s^2+1)\,\mathcal{L}(y) = s + 1$$

$$\mathcal{L}(y) = \frac{s+1}{s^2+1}$$

P.F: $\dfrac{s+1}{s^2+1} = A\dfrac{s}{s^2+1} + B\dfrac{1}{s^2+1}$

$$y = A\cos t + B\sin t$$

$$A = B = 1$$

so $y(t) = \cos t + \sin t$

Example $y'' + 2y' + 5y = 0$ $\quad y(0) = 0$, $y'(0) = 1$

$$y'(0) = 1$$

$$(s^2+2s+5)\mathcal{L}(y) - s\cdot y(0) - y'(0) - 2y(0) = 0$$

$$(s^2+2s+5)\mathcal{L}(y) = 1$$

$$\mathcal{L}(y) = \frac{1}{s^2+2s+5}$$

$$= \frac{1}{(s+1)^2+4} = \frac{1}{(s+1)^2+2^2}$$

(#7 & #8)

$$\begin{cases} \mathcal{L}(e^{at}\sin\beta t) = \dfrac{b}{(s-a)^2+b^2} \\ \mathcal{L}(e^{at}\cos\beta t) = \dfrac{s-a}{(s-a)^2+b^2} \end{cases}$$

$$\frac{1}{(s+1)^2+2^2} = A \cdot \frac{2}{(s+1)^2+2^2} + B\frac{s-1}{(s+1)^2+2^2}$$

$$y(t) = A\,e^{-t}\sin 2t + B\,e^{-t}\cos 2t$$

in our case $A = \frac{1}{2}$ & $B = 0$

$$y(t) = \frac{1}{2}e^{-t}\sin 2t$$

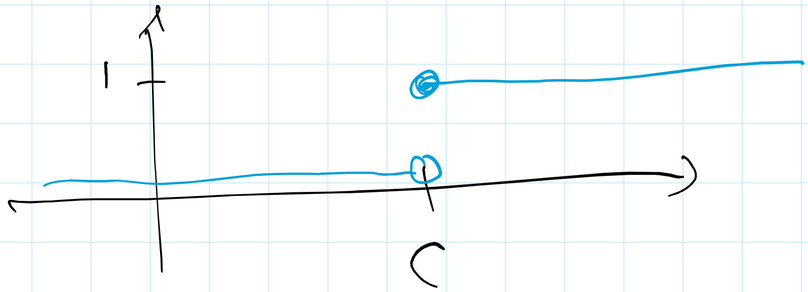
Modified P.F.

$$\ldots\frac{1}{(s-a)}\ldots \rightsquigarrow A\frac{1}{s-a}$$

$$\ldots\frac{1}{(s-a)^2}\ldots \rightsquigarrow A\frac{1}{s-a} + B\frac{1}{(s-a)^2}$$

$$\ldots\frac{1}{(s+a)^2+\beta^2}\ldots \rightsquigarrow A\frac{b}{(s+a)^2+\beta^2} + B\frac{s+a}{(s+a)^2+\beta^2}$$

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \ (?) \end{cases}$$



$u_c(t)$

unit step function

sawtooth

↳ sum of fcts with unit stepfunctions

series method — try to find solutions of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

plug it into the Differential equation and get a formula involving the a_n's.

↳ Applied Analysis II