Multi-Resolution Analysis for the Haar Wavelet

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1 The space $L^2([0,1))$ and its scalar product

We will denote by $L^2([0,1))$ the vector space of all functions $f:[0,1)\to\mathbb{R}$ satisfying[1]

$$\int_0^1 |f(x)|^2\,dx < \infty. \tag{1}$$

On $L^2([0,1))$ one can define a SCALAR PRODUCT as follows:

$$< f,g >= \int_0^1 f(x)\cdot g(x)\,dx. \tag{2}$$

Similarly to the case of $\mathbb{R}^n$, the scalar product automatically defines a NORM on $L^2([0,1))$ via the definition

$$\|f\| = \sqrt{< f,f >} = \sqrt{\int_0^1 |f(x)|^2\,dx} \tag{3}$$

Finally, we say that a sequence (f_n) of functions in $L^2([0,1))$ converges to a function $f(x)\in L^2([0,1))$ IN THE L^2-SENSE, if

$$\lim_{n\to\infty} \|f_n - f\| = 0 \tag{4}$$

2 Orthonormal sets

We say that a set B of elements in $L^2([0,1))$ is an ORTHONORMAL SET, if the scalar product of each element in B with itself equals 1, and the scalar product of two different elements in B is equal to 0:

1. $< f,f >= 1 \quad \text{for all } f\in B$

[1]More precisely: those (equivalence classes of) measurable functions on $[0,1)$ whose square is Lebesgue-integrable. Since we will ultimately be interested in the discrete case anyway, you can just think of bounded piecewise-continuous functions with the Riemann integral.

2. $< f, g >= 0$ for all $f, g \in B$ satisfying $f \neq g$.

An orthonormal set B is automatically linearly independent.[2] Our **ultimate goal** will be to find a particular orthonormal set $B = \{f_1(x), f_2(x), \ldots\}$ such that we can approximate every function $f(x) \in L^2([0,1))$ by linear combinations of the elements in B; more precisely, given $f(x) \in L^2([0,1))$, we will be able to find scalars (a_k) such that the sequence

$$\left(\sum_{k=1}^{n} a_k f_k(x) \right) \tag{5}$$

converges to $f(x)$ in the L^2-sense.[3]

3 The Haar scaling function

We denote by $\phi(x)$ the following function:

$$\phi(x) = \begin{cases} 1, & \text{if } x \in [0,1) \\ 0, & \text{if } x < 0 \text{ or } x \geqslant 1 \end{cases} \tag{6}$$

$\phi : \mathbb{R} \to \mathbb{R}$ is called the Haar scaling function, or the Haar "father" wavelet. Throughout we will identify $\phi(x)$ with its restriction to $[0,1)$.

Let V_0 denote the one-dimensional vector space spanned by $\phi(x)$; this is nothing else but the set of all functions constant on $[0,1)$ (and vanishing elsewhere).

Next we consider the functions $2^{1/2}\phi(2x)$ and $2^{1/2}\phi(2x-1)$. They span a two-dimensional vector space, denoted by V_1, consisting of all functions on $[0,1)$ that are constant both on $[0, \frac{1}{2})$ and on $[\frac{1}{2}, 1)$. Note that $V_0 \subset V_1$.

[2] Indeed assume that for some real numbers $a_1, a_2, \ldots, a_n$ and some distinct elements $g_1, g_2, \ldots, g_n$ in B,

$$a_1 g_1(x) + a_2 g_2(x) + \cdots + a_n g_n(x) = 0 \text{ for all } x.$$

Then for any k with $k \in \{1, \ldots, n\}$

$$< a_1 g_1 + a_2 g_2 + \cdots + a_n g_n, g_k >=< 0, g_k >= 0.$$

On the other hand,

$$< a_1 g_1 + a_2 g_2 + \cdots + a_n g_n, g_k >= a_1 < g_1, g_k > + a_2 < g_2, g_k > + \cdots + a_n < g_n, g_k >= a_k < g_k, g_k >= a_k.$$

So $a_k = 0$ for all k.

[3] We basically already know one example of such a set: It is known that the set

$$F = \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\cos(2x), \ldots, \frac{1}{\sqrt{\pi}}\sin(x), \frac{1}{\sqrt{\pi}}\sin(2x), \ldots\}$$

forms an orthonormal set with which we can approximate all elements in $L^2([-\pi, \pi])$ in this fashion.

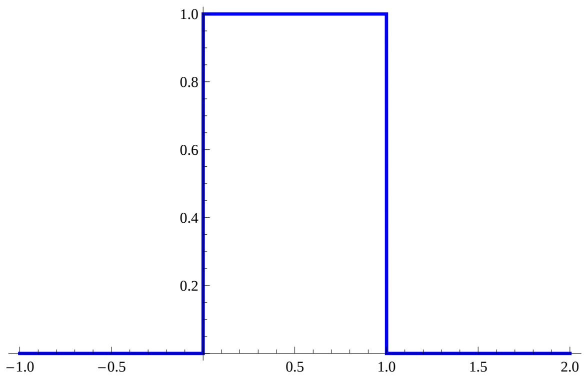

Figure 1: The Haar scaling function $\phi(x)$

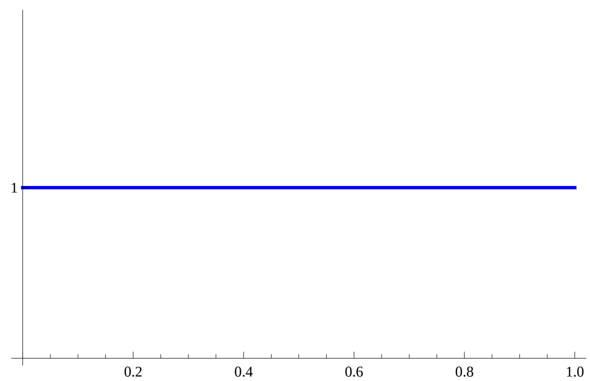

Figure 2: The Haar scaling function $\phi(x)$ restricted to $[0, 1)$

Continuing in this fashion, we can define a 2^j-dimensional vector space V_j, spanned by the functions

$$2^{j/2}\phi(2^j x), 2^{j/2}\phi(2^j x - 1), \ldots, 2^{j/2}\phi(2^j x - (2^j - 1)).$$

The vector space V_j consists of all functions on $[0, 1)$ that are constant on intervals of the form $[k2^{-j}, (k + 1)2^{-j})$ for $k = 0, 1, 2, \ldots 2^j - 1$. Figure 5 shows the function $2^{3/2}\phi(2^3 x - 5)$ contained in V_3. We have $V_0 \subset V_1 \subset \cdots \subset V_j \subset \cdots$.

You should have wondered by now why the factor $2^{j/2}$ is included. The answer is straightforward: this way the functions form an orthonormal set!

Exercise 1

Show that the set $\{2^{j/2}\phi(2^j x), 2^{j/2}\phi(2^j x - 1), \ldots, 2^{j/2}\phi(2^j x - (2^j - 1))\}$ forms an orthonormal set of functions in the vector space V_j.

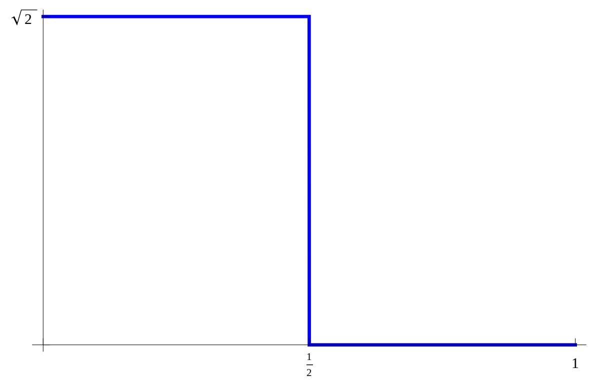

Figure 3: The function $2^{1/2}\phi(2x)$

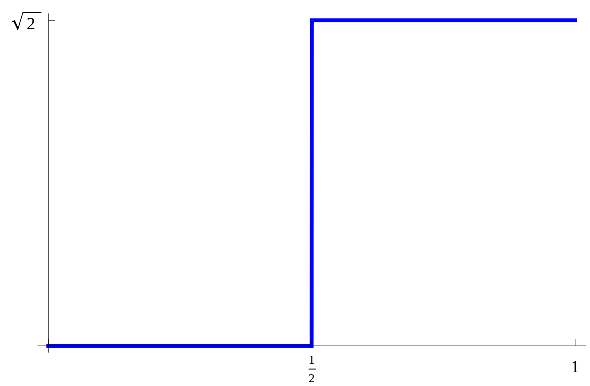

Figure 4: The function $2^{1/2}\phi(2x-1)$

4 Using V_j to approximate functions in $L^2([0,1))$

A function $f \in V_j$ has the form

$$f(x) = a_0 2^{j/2}\phi(2^j x) + a_1 2^{j/2}\phi(2^j x - 1) + \cdots + a_{2^j-1} 2^{j/2}\phi(2^j x - (2^j - 1)). \tag{7}$$

Since the functions on the right side form an orthonormal set, the coefficients a_k are given by the formula

$$a_k = < f(x), 2^{j/2}\phi(2^j x - k) > = \int_0^1 f(x) \cdot 2^{j/2}\phi(2^j x - k)\, dx \tag{8}$$

Exercise 2
Take the scalar product with $2^{j/2}\phi(2^j x - k)$ on both sides of (7) to verify Formula (8).

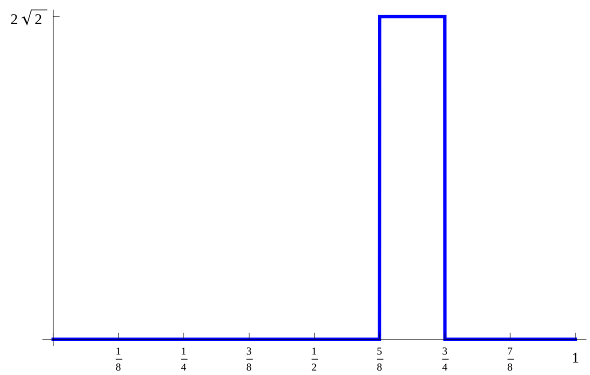

Figure 5: The function $2^{3/2}\phi(2^3x-5)$

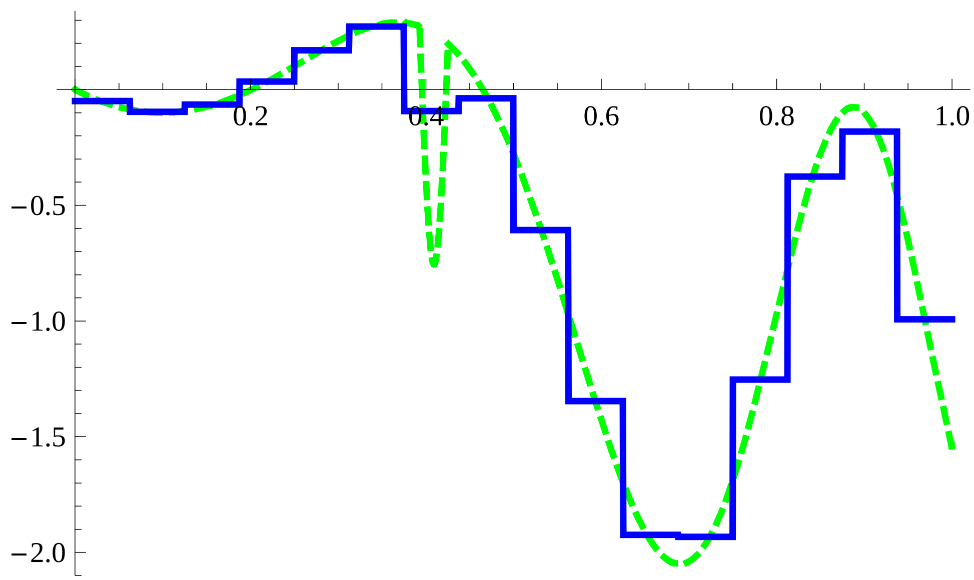

Figure 6: Approximating a function by an element in V_4

The same formula for the coefficients can be used to approximate functions in $L^2([0,1))$ by a function in V_j. Let $f(x)$ be a function in $L^2([0,1))$, and set

$$f_j(x) = a_0 2^{j/2}\phi(2^jx) + a_1 2^{j/2}\phi(2^jx-1) + \cdots + a_{2^j-1}2^{j/2}\phi(2^jx-(2^j-1)), \tag{9}$$

where the coefficients a_k are computed via Formula (8).

Alfred Haar (1885–1933) showed in 1910 that, if $f(x)$ is continuous, the sequence $(f_j(x))$ converges to $f(x)$ uniformly. If, on the other hand, $f(x) \in L^2([0,1))$, then

$$\lim_{j\to\infty} \|f - f_j\| = 0.$$

Figures 6 and 7 show the approximation of a function (dashed line) by an element in V_4 and V_7, respectively (solid line).

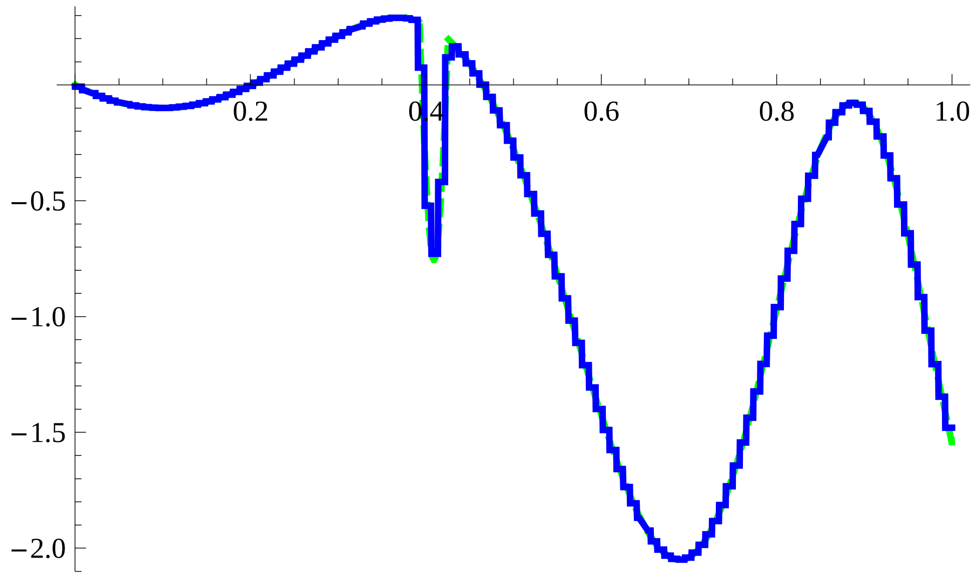

Figure 7: Approximating a function by an element in V_7

While we have found nice orthonormal bases for all the vector spaces V_j, we still fall short of our goal: If we take two basis elements from different V_j's, their scalar product will not necessarily equal zero, because the intervals where the basis elements are equal to 1 may overlap.

5 The Haar wavelet

Let's see whether we can remedy this deficiency step by step. We want to find a function $\psi(x)$ in V_1, such that the linear combinations of $\phi(x)$ and $\psi(x)$ span the vector space V_1, and such that the following conditions are satisfied:

1. $< \phi, \psi >= 0$
2. $< \psi, \psi >= 1$

Since $\psi \in V_1$ we can find scalars a_1 and a_2 such that

$$\psi(x) = a_1\sqrt{2}\phi(2x) + a_2\sqrt{2}\phi(2x-1). \tag{10}$$

Note also that

$$\phi(x) = \phi(2x) + \phi(2x-1). \tag{11}$$

Using (10) and (11), the first condition becomes

$$< \phi, \psi >=< \phi(2x) + \phi(2x-1), a_1\sqrt{2}\phi(2x) + a_2\sqrt{2}\phi(2x-1) >= a_1/\sqrt{2} + a_2/\sqrt{2} = 0, \quad (12)$$

so $a_2 = -a_1$. The second condition yields:

$$< \psi, \psi >=< a_1\sqrt{2}\phi(2x)+a_2\sqrt{2}\phi(2x-1), a_1\sqrt{2}\phi(2x)+a_2\sqrt{2}\phi(2x-1) >= a_1^2+a_2^2 = 1, \quad (13)$$

Solving (12) and (13) for a_1 and a_2, we obtain[4]

$$\psi(x) = \phi(2x) - \phi(2x-1) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}) \\ -1, & \text{if } x \in [\frac{1}{2}, 1) \\ 0, & \text{otherwise} \end{cases} \quad (14)$$

The function $\psi(x)$ is called the HAAR "MOTHER" WAVELET; its graph is depicted in Figure 8.

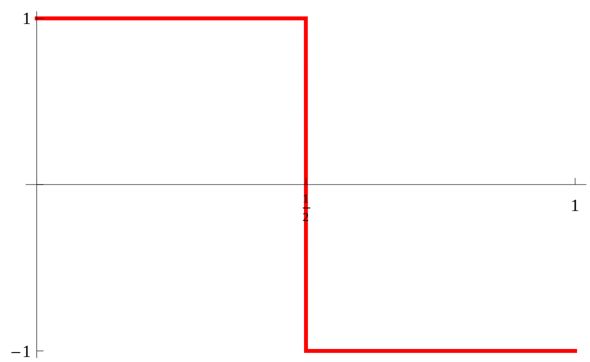

Figure 8: The Haar mother wavelet $\psi(x)$

We will denote the vector space spanned by the function $\psi(x)$ as W_0. It is customary to write

$$V_1 = V_0 \oplus W_0. \quad (15)$$

Here the symbol $\oplus$ is used to indicate that each element in V_1 can be written in a unique way as the sum of an element in V_0 and an element in W_0 and that the scalar product of any element in V_0 with any element in W_0 equals zero.

Exercise 3
Show that the functions $\sqrt{2}\psi(2x)$ and $\sqrt{2}\psi(2x-1)$ are elements in V_2.

[4]There are actually two solutions; it suffices for us to consider the solution for which $a_1 > 0$. Why?

Exercise 4
Show that the set $\{\sqrt{2}\psi(2x),\ \sqrt{2}\psi(2x-1)\}$ forms an orthonormal set.

Exercise 5
Show that $< \sqrt{2}\psi(2x), f(x) >= 0$ for all functions $f(x) \in V_1$. (The same result holds for $\sqrt{2}\psi(2x-1)$.)

These two functions are shown in Figures 9 and 10, respectively.

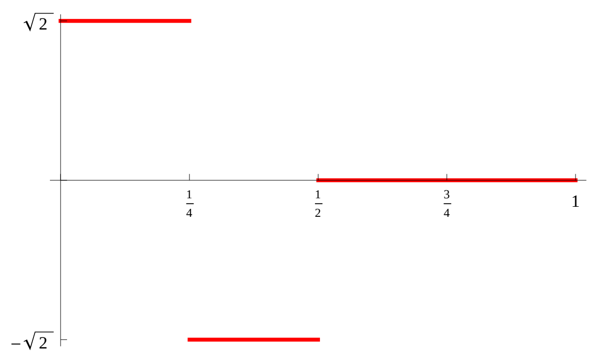

Figure 9: The function $\sqrt{2}\,\psi(2x)$ in W_1

Let's denote the vector space spanned by $\sqrt{2}\psi(2x)$ and $\sqrt{2}\psi(2x-1)$ as W_1. The three exercises above show that

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1, \tag{16}$$

meaning once again that each element in V_2 can be written in a unique way as the sum of an element in V_1 and an element in W_1 and that the scalar product of any element in V_1 with any element in W_1 equals zero.

Continuing in this fashion, we can write

$$V_{j+1} = V_j \oplus W_j, \tag{17}$$

where W_j is the vector space spanned by the functions

$$2^{j/2}\psi(2^j x), 2^{j/2}\psi(2^j x - 1), \ldots, 2^{j/2}\psi(2^j x - (2^j - 1)).$$

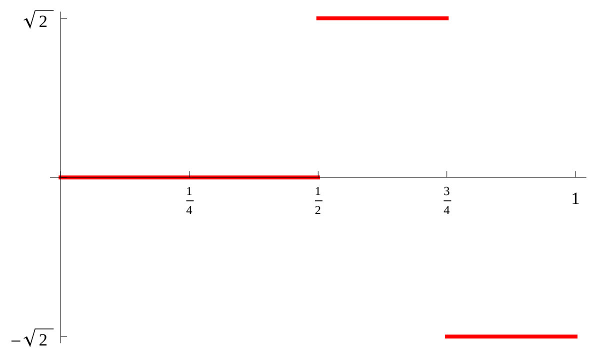

Figure 10: The function $\sqrt{2}\,\psi(2x-1)$ in W_1

The formula

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}$$

is called MULTI-RESOLUTION ANALYSIS.

6 A discrete example

A function in V_4 is determined by its 16 coefficients. Suppose the vector of coefficients is

$$(180, 167, 244, 190, 159, 242, 176, 192, 168, 250, 175, 219, 193, 232, 200, 234) \tag{18}$$

The corresponding function is shown in Figure 11. Note that the coefficients are multiplied by the factor 4 along the way. How can we write this function as a sum of a function in V_3

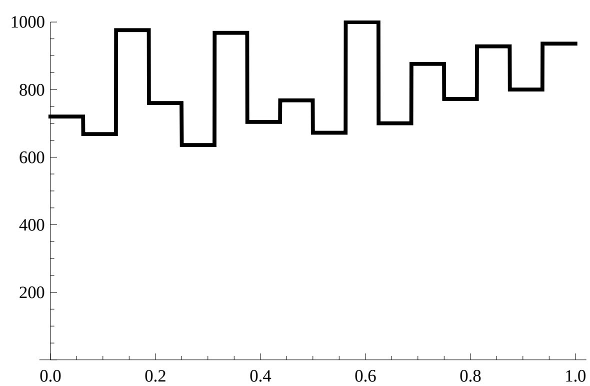

Figure 11: A function in V_4

and a function in W_3? Let's start with the component in V_3. Since for $k \in \{0, 1, \ldots, 7\}$

$$4\phi(16x - 2k) + 4\phi(16x - (2k+1)) = 2^{1/2} \cdot 2^{3/2}\phi(8x - k)), \tag{19}$$

we obtain that the coefficient b_k of the function in V_3 is given for $k \in \{0, 1, \ldots, 7\}$ by

$$b_k = \frac{a_{2k} + a_{2k+1}}{\sqrt{2}}, \tag{20}$$

where a_k denotes the kth coefficient of the function in V_4. In other words, we obtain the vector representing the function in V_3 by multiplying the vector in (18) by the matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{21}$$

In our particular case, the vector (rounded to the nearest integer) representing the function in V_3 is given by

$$(245, 307, 284, 260, 296, 279, 301, 307). \tag{22}$$

Figure 12 shows the function in V_4 and its "blurry" counterpart in V_3.

What about the component of our function in W_3? Since for $k \in \{0, 1, \ldots, 7\}$

$$4\phi(16x - 2k) - 4\phi(16x - (2k+1)) = 2^{1/2} \cdot 2^{3/2}\psi(8x - k)), \tag{23}$$

we obtain that the coefficient c_k of the function in W_3 is given by

$$c_k = \frac{a_{2k} - a_{2k+1}}{\sqrt{2}}, \tag{24}$$

where, again, a_k denotes the kth coefficient of the function in V_4. In other words, this time we obtain the vector representing the function in W_3 by multiplying the vector in (18) by

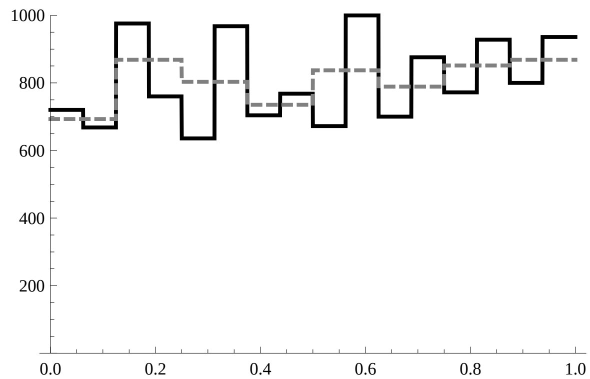

Figure 12: A function in V_4 (black) and its "orthogonal projection" onto V_3 (dashed)

the matrix

$$\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix} \tag{25}$$

In our particular case, the vector (rounded to the nearest integer) representing the function in W_3 is given by

$$(9, 38, -59, -11, -58, -31, -28, -24). \tag{26}$$

Figure 13 shows this as a function in W_3.

If we are "joining" the vectors in (22) and (26), we obtain

$$(245, 307, 284, 260, 296, 279, 301, 307, 9, 38, -59, -11, -58, -31, -28, -24). \tag{27}$$

Since our function in V_4 is the sum of its orthogonal projections onto V_3 and W_3, we will be able to retrieve the vector in (18) from the vector (27). The cumulative energy plots of

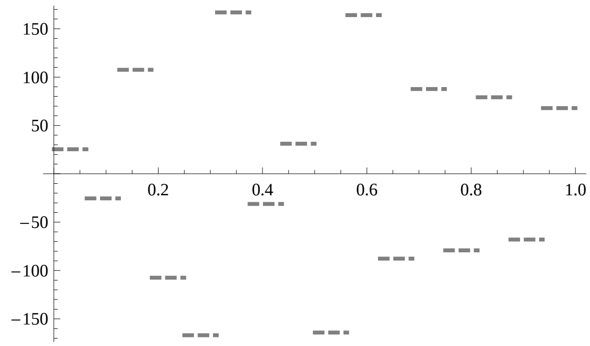

Figure 13: The "orthogonal projection" of our function onto W_3 (dashed)

both vectors are shown in Figure 14, indicating that the vector in (27) has a higher energy concentration than the original vector (18) and thus may be considered as a compressed version of the vector in (18).

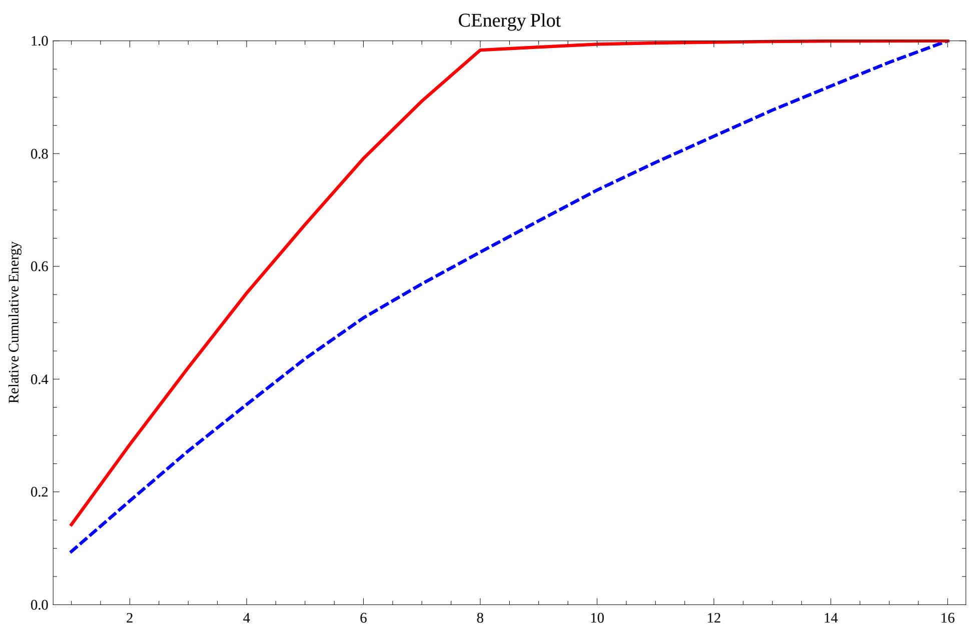

Figure 14: The original vector (18) is shown dashed, while vector (27) is depicted solid.

Exercise 6

This section has shown how to compute vector (27) from vector (18). Can we reverse the procedure? Suppose our procedure produces as the vector in (27)

$$(200, 350, 351, 130, 115, 215, 122, 308, 15, 35, 47, 23, -12, -32, 67, -23).$$

What does the corresponding original vector (18) look like?

7 Concluding Remarks

We have outlined a general procedure: (1) start with a father wavelet $\phi(x)$, (2) construct an increasing sequence of vector spaces

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots,$$

capable of approximating functions in $L^2([0,1))$, (3) construct the corresponding mother wavelet, and (4) ultimately produce a multi-resolution analysis

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}$$

Our choice for $\phi(x)$ was the constant 1. As you will see later in the course, in the 1980's other possible candidates emerged.

Acknowledgment. This exposition is based on material in *A First Course in Wavelets with Fourier Analysis* by Albert Boggess & Francis J. Narcowich.