Isometries of the Complex Plane

HELMUT KNAUST

Department of Mathematical Sciences
The University of Texas at El Paso
El Paso TX 79968-0514

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1. Introduction. Isometries are distance-preserving maps. We will restrict our attention to complex functions: A function $f : \mathbb{C} \to \mathbb{C}$ is called an *isometry* if it satisfies

$$|f(w) - f(z)| = |w - z| \text{ for all } w, z \in \mathbb{C}.$$

It is quite easy to check that the following examples are isometries:

1. For a fixed $z_0 \in \mathbb{C}$, the function given by $f(z) = z + z_0$ *translates* z by z_0.
2. For a fixed $\theta \in \mathbb{R}$, the function $f(z) = ze^{i\theta}$ *rotates* z around the origin by the angle θ.
3. The function $f(z) = \overline{z}$ *reflects* z about the real axis.

Isometries are injective functions. Indeed assume that $f : \mathbb{C} \to \mathbb{C}$ is an isometry, and that $w \neq z$. Then $0 \neq |w - z| = |f(w) - f(z)|$, so $f(w) \neq f(z)$. It is not as easy to prove directly that isometries in the complex plane are also surjective functions, but this fact will follow from our discussion below.

Isometries are used to define congruence in geometry: two geometric figures A and B are called *congruent* to each other, if there is an isometry $f : \mathbb{C} \to \mathbb{C}$ with $f(A) = B$.

2. Classification of isometries in the complex plane. We will first consider isometries with two fixed points:

Lemma. *Let $h : \mathbb{C} \to \mathbb{C}$ be an isometry that satisfies $h(0) = 0$ and $h(1) = 1$. Then $h(z) = z$ or $h(z) = \overline{z}$.*

Proof: Using the fact that $h : \mathbb{C} \to \mathbb{C}$ is an isometry, the two fixed point conditions yield the following equations:

$$|h(z)| = |z| \tag{1}$$
$$|h(z) - 1| = |z - 1| \tag{2}$$

Squaring both sides of (2) yields

$$(h(z) - 1)(\overline{h(z) - 1}) = (z - 1)(\overline{z - 1}).$$

Using the distributive property we can write this as

$$h(z)\overline{h(z)} - h(z) - \overline{h(z)} + 1 = z\overline{z} - z - \overline{z} + 1;$$

in other words

$$|h(z)|^2 - h(z) - \overline{h(z)} + 1 = |z|^2 - z - \overline{z} + 1.$$

Using (1) it follows that

$$h(z) + \overline{h(z)} = z + \overline{z},$$

so $h(z)$ and z have the same real part. Once we know that $\operatorname{Re} h(z) = \operatorname{Re} z$, we conclude from (1) also that $\operatorname{Im} h(z) = \pm \operatorname{Im} z$. Thus $h : \mathbb{C} \to \mathbb{C}$ is given by $h(z) = z$ or by $h(z) = \overline{z}$, as desired. $\square$

Theorem. *Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary isometry. Then there exist $z_0 \in \mathbb{C}$ and $\theta \in \mathbb{R}$ such that $f(z) = z_0 + ze^{i\theta}$ or $f(z) = z_0 + \overline{z}e^{i\theta}$.*

Proof: We set $z_0 = f(0)$, and then define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = f(z) - z_0.$$

The function g satisfies $g(0) = 0$.

Next consider $g(1)$. Since g is an isometry, it follows that

$$|g(1)| = |g(1) - g(0)| = |1 - 0| = 1,$$

so there is a $\theta \in \mathbb{R}$ such that $g(1) = e^{i\theta}$. We define $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) = g(z)e^{-i\theta}.$$

This function satisfies the two conditions $h(0) = 0$ and $h(1) = 1$, and thus, by the Lemma, $h(z) = z$ or $h(z) = \overline{z}$.

Working our way back we see that $g : \mathbb{C} \to \mathbb{C}$ has the form $g(z) = ze^{i\theta}$ or $g(z) = \overline{z}e^{i\theta}$, and finally we indeed obtain that $f : \mathbb{C} \to \mathbb{C}$ is of the form $f(z) = z_0 + ze^{i\theta}$ or $f(z) = z_0 + \overline{z}e^{i\theta}$. $\square$

We will call these isometries *of Type I*, and *of Type II* respectively.

3. Reflections. A Type I isometry is a rotation around the origin followed by a translation; we will show next that a Type II isometry is a reflection about a line through the origin followed by a translation.

Suppose $r : \mathbb{C} \to \mathbb{C}$ is the isometry that reflects each complex number z about the line through the origin with angle of incline α (see Figure 1). Rotating *clockwise* by α moves this line to the real axis.

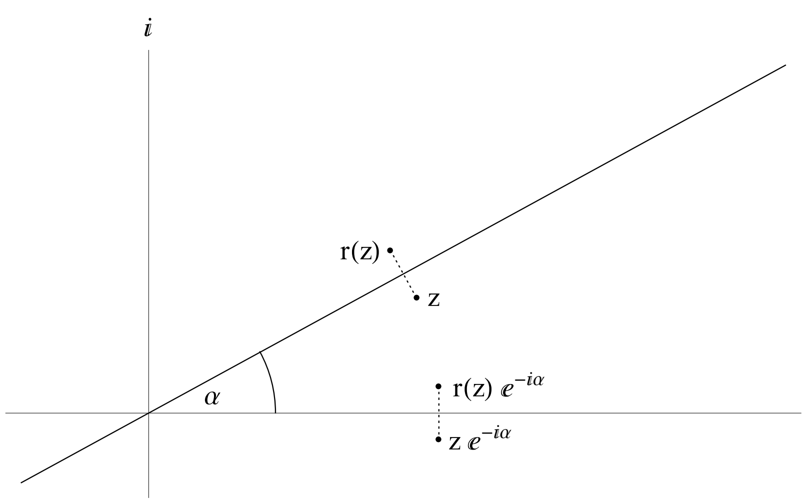

Figure 1: Rotating the line and the points by the angle $-\alpha$.

Consequently $r(z)e^{-i\alpha}$ and $ze^{-i\alpha}$ are a complex conjugate pair:

$$r(z)e^{-i\alpha} = \overline{ze^{-i\alpha}},$$

so we obtain that

$$r(z) = (\overline{z}\ \overline{e^{-i\alpha}})e^{i\alpha} = \overline{z}e^{2i\alpha}.$$

To summarize, reflection about the line through the origin with angle of incline α is reflection about the real axis followed by a rotation about the origin by angle 2α (and vice versa).

4. The group of isometries. The set $\mathcal{D}$ of isometries $f : \mathbb{C} \to \mathbb{C}$ with the binary operation of composition forms a group. In particular, the composition of two Type I isometries (or two Type II isometries) is a Type I isometry, while the composition of a Type I isometry with a Type II isometry yields a Type II isometry. The neutral element of this group is the identity map given by $f(z) = z$. The inverse element of a Type I isometry of the form $f(z) = z_0 + ze^{i\theta}$ is the Type I isometry

$$g(z) = (z - z_0)e^{-i\theta} = -z_0e^{-i\theta} + ze^{-i\theta}.$$

Similarly, a Type II isometry of the form $f(z) = z_0 + \overline{z}e^{i\theta}$ has as its inverse the following Type II isometry:

$$g(z) = \overline{(z - z_0)e^{-i\theta}} = -\overline{z_0}e^{i\theta} + \overline{z}e^{i\theta}.$$

Let us observe that

$$\mathcal{D}_0 = \{f \in \mathcal{D} \mid f(0) = 0\}$$

forms a subgroup of $\mathcal{D}$. Its elements are rotations of the form $f(z) = ze^{i\theta}$ or reflections of the form $f(z) = \overline{z}e^{i\theta}$. Note that isometries $f \in \mathcal{D}_0$ are $\mathbb{R}$-linear maps:

$$f(c_1z_1 + c_2z_2) = c_1f(z_1) + c_2f(z_2) \text{ for all } c_1, c_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{C}.$$

We will see in the next section how to represent these isometries in $\mathcal{D}_0$ as real 2×2 matrices.

5. Isometries in $\mathcal{D}_0$ as orthogonal matrices. Recall that a square matrix A is *orthogonal* if the product with its transpose yields the identity matrix. For 2×2 matrices we obtain the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

or, equivalently, the following three conditions:

$$\begin{aligned} a^2 + b^2 &= 1 \\ c^2 + d^2 &= 1 \\ ac + bd &= 0 \end{aligned}$$

The first two conditions imply that (a, b) and (c, d) both lie on the unit circle; the third condition requires (a, b) and (c, d) to be orthogonal as vectors (see Figure 2). This leaves two possibilities: $(c, d) = (-b, a)$ or $(c, d) = (b, -a)$. Since (a, b) lies on the unit circle, we can also parametrize (a, b) as $(a, b) = (\cos\theta, \sin\theta)$, finally obtaining that all orthogonal 2×2 matrices are of the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{ or } \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

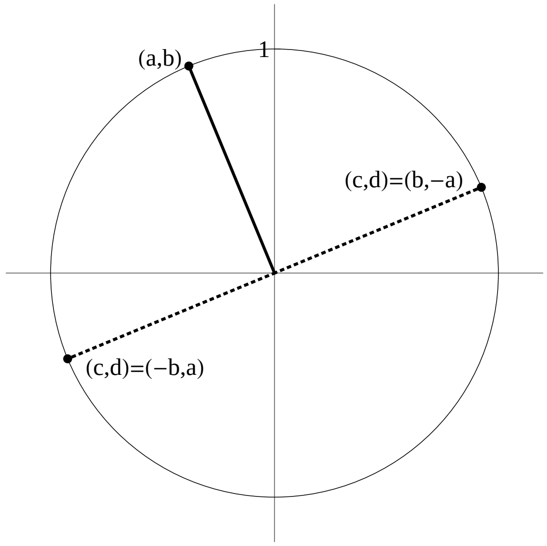

Figure 2: The coefficients of an orthogonal matrix

for some $\theta \in [0, 2\pi)$. Note that $\det\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = 1$, while $\det\left(\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}\right) = -1$. Writing the vector

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

as a complex number, we obtain

$$\begin{aligned} x' + iy' &= (x\cos\theta + y\sin\theta) + i(-x\sin\theta + y\cos\theta) \\ &= x(\cos\theta - i\sin\theta) + iy(\cos\theta - i\sin\theta) \\ &= (x+iy)e^{-i\theta}. \end{aligned}$$

Consequently the orthogonal matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ corresponds to the Type I isometry $f : \mathbb{C} \to \mathbb{C}$, given by

$$f(z) = ze^{-i\theta},$$

and thus rotates a vector *clockwise* about the origin by the angle θ.
A similar computation for

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

yields

$$\begin{aligned} x' + iy' &= (x\cos\theta + y\sin\theta) + i(x\sin\theta - y\cos\theta) \\ &= x(\cos\theta + i\sin\theta) - iy(\cos\theta + i\sin\theta) \\ &= (x-iy)e^{i\theta}. \end{aligned}$$

Thus the orthogonal matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ corresponds to the Type II isometry $f : \mathbb{C} \to \mathbb{C}$, given by

$$f(z) = \overline{z}e^{i\theta}.$$