Boolean Algebra Theorem Sequence

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Statements and their logic connectives on the one hand, and sets and set connectives on the other hand behave somewhat analogously. The English mathematician GEORGE BOOLE *(1815–1864) made this idea precise by describing what he called "algebra of logic"[1]. Today we use the name "Boolean Algebra" in his honor instead. The axioms below were first formulated [2] by the American mathematician* EDWARD V. HUNTINGTON *(1874–1952).*

A *Boolean Algebra* is a set $\mathcal{B}$ together with two "connectives" $\sqcap$ and $\sqcup$ satisfying the following properties:

1. **Closure Laws**:
   (a) If A and B are two elements in $\mathcal{B}$, then $A \sqcap B$ is also an element in $\mathcal{B}$.
   (b) If A and B are two elements in $\mathcal{B}$, then $A \sqcup B$ is also an element in $\mathcal{B}$.
2. **Commutative Laws**:
   (a) $A \sqcap B = B \sqcap A$ for all elements A and B in $\mathcal{B}$.
   (b) $A \sqcup B = B \sqcup A$ for all elements A and B in $\mathcal{B}$.
3. **Distributive Laws**:
   (a) $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C)$ for all elements A, B and C in $\mathcal{B}$.
   (b) $A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C)$ for all elements A, B and C in $\mathcal{B}$.
4. **Identity Laws**:
   There are elements $N \in \mathcal{B}$ (called the *null-element*) and $O \in \mathcal{B}$ (the *one-element*) such that
   (a) $A \sqcap N = N$ and $A \sqcap O = A$ for all elements A in $\mathcal{B}$.
   (b) $A \sqcup O = O$ and $A \sqcup N = A$ for all elements A in $\mathcal{B}$.
5. **Complement Law**:
   For every element A in $\mathcal{B}$ there is an element B in $\mathcal{B}$ such that $A \sqcap B = N$ and $A \sqcup B = O$.
6. **Associative Laws**:[1]
   (a) $A \sqcap (B \sqcap C) = (A \sqcap B) \sqcap C$ for all elements A, B and C in $\mathcal{B}$.
   (b) $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$ for all elements A, B and C in $\mathcal{B}$.

Let X be an arbitrary set. Note that

$$\mathcal{P}(X) = \{A \mid A \subseteq X\},$$

the power set of X, with the connectives $\cap$ (in the role of $\sqcap$) and $\cup$ (in the role of $\sqcup$) forms a Boolean Algebra.

[1] The Associative Laws can be deduced from the other five Boolean Algebra Laws.

Problem 1 Let X be an arbitrary set. Let A and B be elements in $\mathcal{P}(X)$. Show the following

$$A \cap (A \cup B) = A.$$

(Similarly one obtains that $A \cup (A \cap B) = A$.)

Problem 2 Let A and B be elements in a Boolean Algebra $\mathcal{B}$. Show:

$$A \sqcap (A \sqcup B) = A.$$

Analogously one can obtain $A \sqcup (A \sqcap B) = A$.

Problem 3 Let A be an element in a Boolean Algebra $\mathcal{B}$ with null-element N and one-element O. Furthermore let $B, C \in \mathcal{B}$ such that

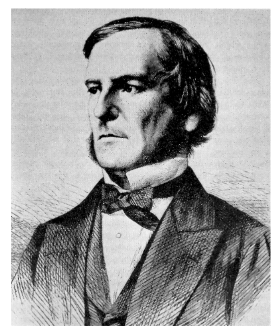
$$A \sqcap B = N \text{ and } A \sqcup B = O,$$

$$A \sqcap C = N \text{ and } A \sqcup C = O.$$

Show that $B = C$.

Certain sets of statements with connectives $\wedge$ (in the role of $\sqcap$) and $\vee$ (in the role of $\sqcup$) also form Boolean Algebras.

What is meant by "certain" sets of statements? Our task at hand is to identify what sets of statements correspond to power sets.



G. Boole

Let us consider an example and start with one "generic" statement P. How many distinct propositional forms can we form involving this statement? A little bit of reflection will lead us on the following path: Every propositional form has a truth table, so the number of distinct propositional forms is limited by the number of distinct truth tables. Since a truth table involving the statement P has two rows, and since we have two choices for each row entry (T or F), there are at most 4 distinct truth tables, and therefore there are at most 4 distinct propositional forms. On the other hand it is easy to see that P, $\neg P$, $P \vee \neg P$ and $P \wedge \neg P$ are 4 distinct propositional forms contained in each Boolean Algebra containing P.

It is now boring to check that the following 4-element set indeed forms a Boolean Algebra:

$$\mathcal{S}_1 = \{P \wedge \neg P;\ P,\ \neg P;\ P \vee \neg P\}$$

$\mathcal{S}_1$ is called the "Boolean Algebra generated by the free statement P".

Problem 4 Find the Boolean Algebra $\mathcal{S}_2$ generated by two free statements P and Q. How many elements does $\mathcal{S}_2$ have?

For a natural number n, let $\mathcal{D}_n$ denote the set of the divisors of n. For example, $\mathcal{D}_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$ and $\mathcal{D}_{12} = \{1, 2, 3, 4, 6, 12\}$. For $m, n \in \mathbb{N}$ let $m \sqcap n$ denote the greatest common divisor of n and m, and $m \sqcup n$ their least common multiple. For instance $6 \sqcap 4 = 2$ and $6 \sqcup 4 = 12$. It turns out that $\mathcal{D}_{42}$ with these two operations $\sqcap$ and $\sqcup$ forms a Boolean Algebra, while $\mathcal{D}_{12}$ does ***not****.*

Problem 5 Verify Boolean Algebra Laws 3, 4 and 5 for $\mathcal{D}_{42}$.

Problem 6

1. Show that $\mathcal{D}_{12}$ does not form a Boolean Algebra.
2. Conjecture for which values of n the set $\mathcal{D}_n$ forms a Boolean Algebra.

Problem 7 Let $\mathcal{B}$ be a Boolean Algebra. Show: $N = O$ if and only if $\mathcal{B} = \{N\}$.

Every Boolean Algebra is endowed with a partial order:

Problem 8 Consider the relation "$\preceq$" on a Boolean Algebra $\mathcal{B}$ defined by

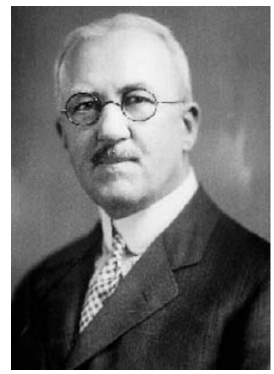
$$A \preceq B \quad \Leftrightarrow \quad A \sqcup B = B$$

for $A, B \in \mathcal{B}$. Prove that $\preceq$ is reflexive, anti-symmetric and transitive.

Problem 9 Show: $A \preceq B \;\Leftrightarrow\; A \sqcap B = A$ for $A, B \in \mathcal{B}$.

Problem 10 Draw *Hasse diagrams* for the Boolean Algebras $\mathcal{S}_1$, and $\mathcal{S}_2$, respectively, endowed with the partial order $\preceq$.

Let X be a set, partially ordered by $\preceq$. We say that $x \in X$ is an IMMEDIATE PREDECESSOR *of $y \in X$ if*
(1) $x \preceq y$, and (2) for all $z \in X$ with $x \preceq z \preceq y$, it follows that $z = x$ or $z = y$.



E.V. HUNTINGTON

Problem 11 Let $\mathcal{B}$ be a Boolean Algebra with null-element N, partially ordered by $\preceq$. We say that $A \in \mathcal{B}$ is an ATOM of $\mathcal{B}$ if N is an immediate predecessor of A.

1. Find all atoms of $\mathcal{P}(\{1, 2, 3, 4\})$.
2. Find all atoms of $\mathcal{D}_{42}$.

Problem 12 Find a Boolean Algebra with 8 elements that is a subset of $\mathcal{P}(\{1, 2, 3, 4\})$, but **not** the power set of a three-element subset of $\{1, 2, 3, 4\}$, then find its atoms and draw its Hasse diagram.

Problem 13 Assume that $\mathcal{B}$ is a Boolean Algebra with finitely many elements. Show that for every $B \in \mathcal{B}$ with $B \neq N$ there is an atom A such that $A \preceq B$.

Problem 14 Let A, B be two elements in a Boolean Algebra. Show the following: If $A \sqcap B = N$ and $A \preceq B$, then $A = N$.

Problem 15 Let A_1, A_2 be two atoms in a finite Boolean Algebra. Show the following:

1. The least upper bound of the set $\{A_1, A_2\}$ is the element $A_1 \sqcup A_2$.
2. If $A_1 \neq A_2$, then $A_1 \sqcap A_2 = N$.

Problem 16 Given an element B in a finite Boolean Algebra $\mathcal{B}$, we let

$$\alpha(B) = \{A \in \mathcal{B} \mid A \preceq B \text{ and } A \text{ is an atom of } \mathcal{B}\}.$$

Let $A_1 \neq A_2$ be two atoms in $\mathcal{B}$. Show that $\alpha(A_1 \sqcup A_2) = \{A_1, A_2\}$.

In the sequel, you may assume that results corresponding to those proved for two atoms in Problems 15 and 16 also hold for finitely many atoms.

Problem 17 Let $B \neq N$ be an element in a finite Boolean Algebra $\mathcal{B}$, and suppose $\alpha(B) = \{A_1, A_2, A_3, \ldots, A_k\}$ for some $k \in \mathbb{N}$ and some atoms $A_1, A_2, A_3, \ldots, A_k$ of $\mathcal{B}$. Show:[2]

$$B = A_1 \sqcup A_2 \sqcup A_3 \sqcup \ldots \sqcup A_k.$$

[2] *Hint:* Expect to use Boolean Algebra Law 5 along the way.

M.H. Stone

The next problem is the finite version of a general representation theorem for Boolean Algebras [3], proved by the American mathematician Marshall H. Stone *(1903–1989).*

Problem 18 Let $\mathcal{B}$ be a finite Boolean Algebra with k atoms for some $k \in \mathbb{N}$, and let $\mathcal{P}(A)$ denote the power set of the set A of all atoms of $\mathcal{B}$.

1. Show that the function $\alpha : \mathcal{B} \to \mathcal{P}(A)$, defined in Problem 16, is a bijection. Thus $\mathcal{B}$ has 2^k elements.
2. Show that the identities $\alpha(B \sqcup B') = \alpha(B) \cup \alpha(B')$ and $\alpha(B \sqcap B') = \alpha(B) \cap \alpha(B')$ hold for all $B, B' \in \mathcal{B}$.

   Additionally, show that $\alpha(N) = \emptyset$ and $\alpha(O) = A$.

The result in Problem 18 is no longer true for infinite Boolean Algebras:

Problem 19

1. Let $\mathcal{B} = \{B \in \mathcal{P}(\mathbb{N}) \mid (B \text{ is finite}) \text{ or } (\mathbb{N} \setminus B \text{ is finite})\}$. Show that $\mathcal{B}$ is a Boolean Algebra (with $\cup$ and $\cap$).
2. Let $\mathcal{A}$ denote the power set of all atoms of $\mathcal{B}$. Show that $\mathcal{B}$ is countably infinite, while $\mathcal{A}$ is uncountable. Therefore the function $\alpha : \mathcal{B} \to \mathcal{A}$, defined in Problem 16, is **not** a bijection.

References:

[1] George Boole: *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities* (1854).

[2] Edward V. Huntington: *Sets of Independent Postulates for the Algebra of Logic*. Transactions of the American Mathematical Society 5 (1904), pp. 288-309.

[3] Marshall H. Stone: *The Theory of Representation for Boolean Algebras*. Transactions of the American Mathematical Society 40 (1936), pp. 37-111.