The following lemma is the crucial ingredient in the proof of Theorem 5.6 in *Gaughan, Introduction to Analysis*.

Lemma *Let $f : [a, b] \to \mathbb{R}$ be a bounded function with $|f(x)| \leq M$ for all $x \in [a, b]$. Let $\epsilon > 0$ be given, and let P_0 be a partition of $[a, b]$ with $n + 1$ elements. Let $\delta = \dfrac{\epsilon}{8Mn}$. Then for any partition P of $[a, b]$ with mesh $\mu(P) < \delta$*

$$\mathcal{U}(P) < \mathcal{U}(P \cup P_0) + \epsilon/4 \qquad \text{and} \qquad \mathcal{L}(P) > \mathcal{L}(P \cup P_0) - \epsilon/4.$$

Proof: We will only prove the first inequality; the proof of the second inequality is analogous. Let $\epsilon > 0$ be given, and let partitions P_0 and $P = \{z_0, z_1, \ldots, z_k\}$ be given as in the statement of the lemma.

For the remainder of the proof we will call an interval $[z_{i-1}, z_i]$ *infected*, if $(z_{i-1}, z_i) \cap P_0 \neq \emptyset$. Note that at most n intervals are infected!

Let $M_i = \sup\{f(t) : \; t \in [z_{i-1}, z_i]\}$ for $i = 1, \ldots, k$. On *non-infected* intervals, the "contributions" towards the upper sum with respect to P and $P \cup P_0$ are the same, namely $M_i \cdot (z_i - z_{i-1})$.

Let us now consider an *infected* interval $[z_{i-1}, z_i]$. Let $(z_{i-1}, z_i) \cap P_0 = \{x^i_1, \ldots, x^i_{m_i - 1}\}$. Set $x^i_0 = z_{i-1}$ and $x^i_{m_i} = z_i$. We define $N^i_j = \sup\{f(t) : \; t \in [x^i_{j-1}, x^i_j]\}$ for $j = 1, \ldots, m_i$. The contribution towards the upper sum with respect to P is still given by $M_i \cdot (z_i - z_{i-1})$, while the contribution towards the upper sum with respect to the finer partition $P \cup P_0$ is now given by

$$\sum_{j=1}^{m_i} N_j \cdot (x^i_j - x^i_{j-1}).$$

Note that we have the "trivial" estimate

$$M_i \leq N^i_j + 2M \text{ for all } j = 1, \ldots, m_i.$$

Let $A = \{i : \; [z_{i-1}, z_i] \text{ is non-infected}\}$ and let $B = \{i : \; [z_{i-1}, z_i] \text{ is infected}\}$. We are ready to deduce the desired estimate:

$$\begin{aligned}
\mathcal{U}(P) &= \sum_{i=1}^{k} M_i(z_i - z_{i-1}) \\
&= \sum_{i \in A} M_i(z_i - z_{i-1}) + \sum_{i \in B} M_i(z_i - z_{i-1}) \\
&\leq \sum_{i \in A} M_i(z_i - z_{i-1}) + \sum_{i \in B} \sum_{j=1}^{m_i} (N^i_j + 2M) \cdot (x^i_j - x^i_{j-1}) \\
&< \sum_{i \in A} M_i(z_i - z_{i-1}) + \sum_{i \in B} \left(\sum_{j=1}^{m_i} N^i_j \cdot (x^i_j - x^i_{j-1}) + 2M\delta \right) \\
&\leq \sum_{i \in A} M_i(z_i - z_{i-1}) + \left(\sum_{i \in B} \sum_{j=1}^{m_i} (N^i_j \cdot (x^i_j - x^i_{j-1}) \right) + 2M\delta n \\
&\leq \mathcal{U}(P \cup P_0) + \epsilon/4.
\end{aligned}$$