Practice Final | **Analysis** | **Fall 2000**

No books, notes etc. are allowed.

Problem 1. For which $x \in \mathbb{R}$ does the series $1+2x+3x^2+4x^3+5x^4+6x^5+\ldots$ converge? Find the sum of the series, where it converges.

Problem 2. **(a)** Give the definition of compactness for a metric space.

(b) Show: If $f : X \to Y$ is a continuous function between two metric spaces X and Y, and $A \subseteq X$ is compact, then $f(A)$ is compact.

Problem 3. **(a)** Let A be a subset of a metric space (X, d). Define the concept "A is connected in X".

(b) Prove or give a counterexample: If $f : X \to Y$ is a continuous function between two metric spaces X and Y, and $B \subseteq Y$ is connected, then $f^{-1}(B)$ is connected.

Problem 4. **(a)** Define the notion of uniform convergence of a sequence of functions.

(b) Let $f_n, f : X \to Y$ be functions between two metric spaces X and Y. Show: if f_n is continuous on X for all $n \in \mathbb{N}$, and (f_n) converges to f uniformly on X, then f is continuous on X.

Problem 5. Show: If X is a *compact* metric space and $f : X \to X$ is a function satisfying

$$d(f(x), f(y)) < d(x, y) \text{ for all } x \neq y \in X,$$

then f has a unique fixed point on X. A fixed point of a function f is a point $x \in X$ such that $f(x) = x$.
Hint: Does the set $\{d(x, f(x)) \mid x \in X\}$ attain its minimum?

Problem 6. **(a)** State the Weierstrass Approximation Theorem.

(b) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function so that $\int_0^1 x^n \cdot f(x)\, dx = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Use the Weierstrass Approximation Theorem to prove that $f(x) = 0$ for all $x \in [0, 1]$.