The problems are due on Thursday, September 21.

For all students:

Problem 1. Show that \( \mathbb{R}^n \) endowed with
\[
\|(x_1, x_2, \ldots, x_n)\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}
\]
is a normed vector space!

Problem 2. Consider the following vector spaces (with pointwise addition and scalar multiplication):
\[C = \text{the set of continuous functions on } [0, 1],\]
\[R = \text{the set of Riemann-integrable functions on } [0, 1],\]
\[D = \text{the set of functions } f \text{ on } [0, 1], \text{ which are of the form } f = g - h, \text{ where } g, h \text{ are continuous and non-decreasing on } [0, 1].\]

Let \( ||f|| = \sup_{x \in [0,1]} |f(x)| \). Which of the 3 spaces are normed vector spaces under \( || \cdot || \)?

Problem 3. Consider \( C \), endowed with the sup-norm \( \|f\| = \sup_{x \in [0,1]} |f(x)| \) (see Problem 2). Let
\[A = \{ f \in C \mid f(1/2) = 0 \}\]
Is \( A \) open, closed? Find the interior, the closure and the boundary of \( A \).

For graduate students:

Problem 1G. Recall the Bolzano-Weierstrass Theorem: Every bounded infinite subset of real numbers has an accumulation point.

Give an example to show that this result does not generalize to arbitrary metric spaces. (Remark: A subset \( A \) of a metric space \( (X, d) \) is called bounded, if there is an \( x \in X \) and \( R > 0 \) such that \( A \subseteq D(x, R) \).)

Problem 2G. 1.) Show that \( A \subseteq \mathbb{R} \) is open iff \( A \) is the countable union of disjoint open intervals.

2.) Is every closed subset of \( \mathbb{R} \) the countable intersection of closed intervals?

3.) Is a subset \( A \) in an arbitrary metric space \( (X, d) \) open iff \( A \) is the countable union of open balls?