Homework 1 **Analysis** **Fall 2000**

The problems are due on Thursday, September 21.

For all students:

Problem 1. Show that $\mathbb{R}^n$ endowed with

$$\|(x_1, x_2, \ldots, x_n)\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

is a normed vector space!

Problem 2. Consider the following vector spaces (with pointwise addition and scalar multiplication):
$C =$ the set of continuous functions on $[0, 1]$.
$R =$ the set of Riemann-integrable functions on $[0, 1]$.
$D =$ the set of functions f on $[0, 1]$, which are of the form $f = g - h$, where g, h are continuous and non-decreasing on $[0, 1]$.

Let $||f|| = \sup_{x\in[0,1]} |f(x)|$. Which of the 3 spaces are normed vector spaces under $|| \cdot ||$?

Problem 3. Consider C, endowed with the sup-norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$ (see Problem 2). Let

$$A = \{f \in C \mid f(1/2) = 0\}$$

Is A open, closed? Find the interior, the closure and the boundary of A.

For graduate students:

Problem 1G. Recall the Bolzano-Weierstrass Theorem: *Every bounded infinite subset of real numbers has an accumulation point.*

Give an example to show that this result does not generalize to arbitrary metric spaces. (Remark: A subset A of a metric space (X, d) is called *bounded*, if there is an $x \in X$ and $R > 0$ such that $A \subseteq D(x, R)$.)

Problem 2G. 1.) Show that $A \subseteq \mathbb{R}$ is open iff A is the countable union of disjoint open intervals.

2.) Is every closed subset of $\mathbb{R}$ the countable intersection of closed intervals?

3.) Is a subset A in an arbitrary metric space (X, d) open iff A is the countable union of open balls?