Homework 2 **Analysis** **Fall 2000**

The problems are due on Thursday, October 5.

For all students:

Problem 1. Let A be a set in a metric space (X, d), and let $x \notin A$. Show that x is an accumulation point of A if and only if there is a sequence (x_n) of elements in A converging to x.

Problem 2. Consider the following two binary operations on $(0, 1]$:

$$d(x, y) = |x - y| \text{ for all } x, y \in (0, 1],$$

$$d^*(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right| \text{ for all } x, y \in (0, 1].$$

1. Show that d and d^* are both metrics on $(0, 1]$.
2. Show that both metrics define the same open sets.
3. Show that d^* is complete, while d is *not* complete.

Problem 3. Let (x_n) and (y_n) be two Cauchy sequences in a metric space (X, d). Show that the sequence $(d(x_n, y_n))$ converges.

For graduate students:

A metric space (X, d) is called *separable* if it contains a countable dense subset. (The real numbers are separable, since the rational numbers form a countable dense subset.)

Problem 1G. Let $(\mathbb{R}, |\,.\,|)$ be the usual metric space of real numbers. Show that $(\mathbb{R} \setminus \mathbb{Q}, |\,.\,|)$ is separable.

Problem 2G. Generalize as follows: If (X, d) is a separable metric space, and if $Y \subseteq X$, then (Y, d) is separable.