Homework 4 **Analysis** **Fall 2000**

The problems are due on Thursday, November 9.

For all students:

Problem 1. A set A in a metric space (X, d) is called an F_σ*-set*, if A is the countable union of closed sets. Example: $\mathbb{Q}$ is an F_σ-set in $\mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $A \subseteq \mathbb{R}$ be an F_σ-set. Show that $f(A)$ is also an F_σ-set.

Does this result easily extend to arbitrary metric spaces? Explain!

Problem 2. Let (X, d) be a compact metric space, and let $f : X \to X$ be a function satisfying

$$d(f(x), f(y)) < d(x, y) \text{ for all } x \neq y,\ x, y \in X.$$

Show that there is a $z \in X$ with $f(z) = z$.
Hint: Consider $\inf\{d(x, f(x)) \mid x \in X\}$.

Problem 3. Let (X, d) be a connected metric space with at least two elements. Show that X is uncountable.

For graduate students:

Problem 1G. Let A be a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 \setminus A$ is path-connected.

Problem 2G. Let $f : A \to \mathbb{R}$ be a continuous function, where $A \subseteq \mathbb{R}$ is a closed set. Show that there is a continuous function $g : \mathbb{R} \to \mathbb{R}$, such that $g(x) = f(x)$ for all $x \in A$.