Homework 5 **Analysis** **Fall 2000**

The problems are due on Tuesday, November 28.

For all students:

Problem 1. For a real number x, define $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

Let (a_n) be a sequence of real numbers, such that $\sum a_n$ converges, while $\sum |a_n|$ diverges.

1. Show that the partial sums of $\sum a_n^+$ and the partial sums of $\sum a_n^-$ are unbounded increasing sequences. Note that $\lim_{n\to\infty} a_n^+ = 0$ and $\lim_{n\to\infty} a_n^- = 0$.
2. Let $L \in \mathbb{R}$. Using the results above, show that there is a bijection $\rho : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\rho(n)} = L.$$

Problem 2. Let (a_n) be a sequence of real numbers, such that $\sum |a_n|$ converges, and let $\rho : \mathbb{N} \to \mathbb{N}$ be any bijection. Show that

$$\sum_{n=1}^{\infty} a_{\rho(n)} = \sum_{n=1}^{\infty} a_n.$$

Problem 3.

1. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. Show that

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

2. Explain why this shows that the root test is "stronger" than the ratio test.

For graduate students:

Problem 1+2G. Let $(f_n) : \mathbb{R} \to [0, 1]$ be a sequence of *increasing* functions, i.e. satisfying $f_n(x) \le f_n(y)$ for all $x \le y$ and all $n \in \mathbb{N}$. The problem will establish that a subsequence of (f_n) converges pointwise on $\mathbb{R}$.

1. Show that there is a subsequence (f_{n_k}) of (f_n), which converges at all rational points in $\mathbb{R}$, say $\lim\limits_{k\to\infty} f_{n_k}(q) =: f(q)$ for all rational numbers q.
2. For $x \in \mathbb{R}$ define $f(x) := \sup\{f(q) \mid q \le x\ ,\ q \text{ is rational}\}$. Using the monotonicity of the functions involved, show that $f(x) = \lim\limits_{k\to\infty} f_{n_k}(x)$ for all $x \in \mathbb{R}$, at which f is continuous.
3. Show that a further subsequence of (f_{n_k}) converges to f for *all* $x \in \mathbb{R}$.
   Hint: How many points are there, at which f is discontinuous?