Homework 2 **Introduction to Analysis** **Spring 2001**

The problems are due on April 2.

Problem 1. Let A be a set of real numbers, and let B denote the set of accumulation points of A. Show that $A \cup B$ contains all of its accumulation points.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$ so that both $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$. Show that f is uniformly continuous on $\mathbb{R}$.

(We say that $\lim_{x\to\infty} f(x) = L$ if for all $\epsilon > 0$ there is an $x_0 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$:

$$(x > x_0 \Rightarrow |f(x) - L| < \epsilon) .$$

The limit at $-\infty$ is defined analogously.)

Problem 3. Let $f : D \to \mathbb{R}$ be a function, and let x_0 be in $\mathbb{R}$. Let $r : D \cap (x_0, \infty) \to \mathbb{R}$ be the restriction of f to $D \cap (x_0, \infty)$, *i.e.* $r(x) := f(x)$ for all $x \in D$ with $x > x_0$. If x_0 is an accumulation point of $D \cap (x_0, \infty)$, we say that f has a *right-hand limit* at x_0 (denoted by $\lim_{x\to x_0^+} f(x)$), if $\lim_{x\to x_0} r(x)$ exists.

Similarly we define $l : D \cap (-\infty, x_0) \to \mathbb{R}$ to be the restriction of f to $D \cap (-\infty, x_0)$. Let x_0 be an accumulation point of $D \cap (-\infty, x_0)$. Then f is said to have a *left-hand limit* at x_0 (denoted by $\lim_{x\to x_0^-} f(x)$), if $\lim_{x\to x_0} l(x)$ exists.

a. Let $f : D \to \mathbb{R}$, and let x_0 be an accumulation point of both $D \cap (x_0, \infty)$ and of $D \cap (-\infty, x_0)$. Show that $\lim_{x\to x_0} f(x)$ exists if and only if both $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ exist and are equal.

b. Give an example of a function for which the right-hand limit at a point does not exist.

c. Show that for an increasing function $f : [a, b] \to \mathbb{R}$ the left-hand limit exists for all $x \in (a, b]$.