Homework 3 **Introduction to Analysis** **Spring 2001**

The problems are due on April 30.

The first two problems explore the connections between the following properties of a function $f : [a, b] \to \mathbb{R}$.

1. The function f is said to have a *bounded derivative*, if f is differentiable on $[a, b]$, and if there is a $M \geq 0$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$.
2. The function f is said to be a *Lipschitz function*, if there is a $M \geq 0$, such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.
3. f is uniformly continuous on $[a, b]$.

Problem 1. Show that 1. $\Rightarrow$ 2. $\Rightarrow$ 3.

Problem 2. Give examples to show that 3. $\not\Rightarrow$ 2. $\not\Rightarrow$ 1.

Problem 3. Consider the function

$$f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & , \text{if } x \neq 0 \\ 0 & , \text{if } x = 0 \end{cases}$$

Show that there is a number $z \in \mathbb{R}$ such that $f'(z) > 0$, but there does not exist a neighborhood of z in which f is increasing.