Homework 5 **Analysis** **Spring 2001**

The problems are due on Wednesday, April 18.

Problem 1. A set A in a metric space (X, d) is called an *F_σ-set*, if A is the countable union of closed sets. Example: $\mathbb{Q}$ is an F_σ-set in $\mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $A \subseteq \mathbb{R}$ be an F_σ-set. Show that $f(A)$ is also an F_σ-set.

Does this result easily extend to arbitrary metric spaces? Explain!

Problem 2. For a real number x, define $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

Let (a_n) be a sequence of real numbers, such that $\sum a_n$ converges, while $\sum |a_n|$ diverges.

1. Show that the partial sums of $\sum a_n^+$ and the partial sums of $\sum a_n^-$ are unbounded increasing sequences. Note that $\lim_{n\to\infty} a_n^+ = 0$ and $\lim_{n\to\infty} a_n^- = 0$.

2. Let $L \in \mathbb{R}$. Using the results above, show that there is a bijection $\rho : \mathbb{N} \to \mathbb{N}$ such that
$$\sum_{n=1}^{\infty} a_{\rho(n)} = L.$$

Problem 3. Show: If $f : X \to Y$ is uniformly continuous, and (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is a Cauchy sequence in Y.

Is this result also true, if one only assumes that f is continuous?