**Homework 2** **Analysis** **Fall 2001**

*The problems are due on Thursday, October 4.*

**For all students:**

**Problem 1.** Let $A$ be a set in a metric space $(X, d)$, and let $x \notin A$. Show that $x$ is an accumulation point of A if and only if there is a sequence $(x_n)$ of elements in $A$ converging to $x$.

**Problem 2.** Consider the following two binary operations on $(0, 1]$:

$$d(x, y) = |x - y| \text{ for all } x, y \in (0, 1],$$

$$d^*(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| \text{ for all } x, y \in (0, 1].$$

1. Show that $d^*$ is a metric on $(0, 1]$.
2. Show that both metrics define the same open sets.
3. Show that $d^*$ is complete, while $d$ is *not* complete.

**Problem 3.** Let $(x_n)$ and $(y_n)$ be two Cauchy sequences in a metric space $(X, d)$. Show that the sequence $(d(x_n, y_n))$ converges.

**For graduate students:**

*Let's denote the set of all Cauchy sequences of rational numbers by $\mathcal{C}$. We say that two Cauchy sequences $(a_n)$ and $(b_n)$ of rational numbers are* **equivalent**, *if $\lim_{n\to\infty}(a_n - b_n) = 0$. If two Cauchy sequences $(a_n)$ and $(b_n)$ are equivalent, we write $(a_n) \sim (b_n)$. (Observe that $\sim$ is indeed an equivalence relation!)*

*An equivalence class $[(a_n)]$ is the set of all Cauchy sequences of rational numbers equivalent to $(a_n)$:*

$$[(a_n)] := \{(b_n) \in \mathcal{C} \mid (b_n) \sim (a_n)\}$$

*Note that $[(a_n)] = [(b_n)]$ if and only if $(a_n) \sim (b_n)$.*

*We denote the set of all such equivalence classes by $\mathcal{R}$. $\mathcal{R}$ can be considered as a* **model** *for the set of real numbers. (To every equivalence class in $\mathcal{R}$ there corresponds in a unique way a real number: The real number associated with $[(a_n)] \in \mathcal{R}$ is its limit $\lim_{n\to\infty} a_n$. In particular, an equivalence class $[(a_n)]$ represents a rational number if and only if $(a_n)$ is equivalent to a constant sequence.)*

*From now on we will call the elements of $\mathcal{R}$* **real numbers**.

*We define addition $\oplus$ and multiplication $\otimes$ of real numbers as follows:*

$$[(a_n)] \oplus [(b_n)] := [(a_n + b_n)]; \quad [(a_n)] \otimes [(b_n)] := [(a_n \cdot b_n)]$$

*Subtraction $\ominus$ and division $\oslash$ are defined similarly.*

**Problem 1G.**

1. Show that the addition $\oplus$ is well-defined. (You have to show two things. First establish that the sum of two Cauchy sequences is Cauchy, then show: if $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$, then $[(a_n)] \oplus [(b_n)] = [(a'_n)] \oplus [(b'_n)]$.)

2. One can similarly show that the multiplication $\otimes$ is well-defined. Show that the multiplication $\otimes$ is commutative and associative. Find the neutral element with respect to multiplication in $\mathcal{R}$.

3. Show the distributive law

$$([(a_n)] \oplus [(b_n)]) \otimes [(c_n)] = ([(a_n)] \otimes [(c_n)]) \oplus ([(b_n)] \otimes [(c_n)])\,.$$

*We say $[(a_n)]$ is positive, if there are a rational number $\epsilon > 0$ and $N \in \mathbb{N}$ so that $a_n > \epsilon$ for all $n \geq N$. We then define an order $\prec$ on $\mathcal{R}$ as follows:*

$$[(a_n)] \prec [(b_n)] \Leftrightarrow [(b_n)] \ominus [(a_n)] \text{ is positive.}$$

**Problem 2G.**

1. Show that the order $\prec$ is well-defined.

2. Show that the order $\prec$ is transitive.

3. Show that $[(a_n)] \prec [(b_n)]$ implies $[(a_n)] \oplus [(c_n)] \prec [(b_n)] \oplus [(c_n)]$ for all $[(a_n)], [(b_n)]$ and $[(c_n)] \in \mathcal{R}$.

4. Show for any real number $[(a_n)]$: Either $[(a_n)]$ is positive, $[(-a_n)]$ is positive, or $[(a_n)]$ is the equivalence class of the sequence, which is constantly 0.

*By now, you have established about half of the axioms defining an ordered field, and in fact, $\mathcal{R}$ is an ordered field. What about the completeness axiom? With considerably more effort and using an appropriate set of axioms for the natural numbers, one can indeed show that with this definition the real numbers satisfy the completeness axiom.*