Homework 3 **Analysis** **Fall 2001**

The problems are due on Tuesday, October 23.

For all students:

Problem 1. Let (A_n) be a sequence of non-empty closed bounded subsets in a complete metric space (X, d), such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, and such that $\lim_{n\to\infty} \operatorname{diam} A_n = 0$. Show that $\bigcap_{n=1}^{\infty} A_n$ consists of exactly one point.

Recall that the diameter of a bounded set A in a metric space is defined as: $\operatorname{diam} A = \sup\{d(x, y) \mid x, y \in A\}$.

Problem 2. A metric space (X, d) is called *countably compact*, if every open cover of X, which consists of **countably** many open sets, contains a finite subcover. Show that every countably compact metric space is (sequentially) compact.

Problem 3. Consider C, the normed vector space of continuous functions on the interval $[0, 1]$, endowed with the sup-norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Show that $B = \{f \in C \mid \|f\| \leq 1\}$ fails to be compact.

Hint: Find a seqence (f_n) in B which satisfies $\|f_n - f_m\| \geq 1$ for all $n \neq m$.

For graduate students:

Problem 1G. Show that every compact metric space is separable. (A metric space is called *separable*, if it contains a countable dense subset.)

Problem 2G. Let (X, d) be a bounded metric space, and denote by F_X the set of all closed non-empty subsets of X. Define for $A, B \in F_X$

$$\rho(A, B) = \sup_{x\in A} \left(\inf_{y\in B} d(x, y) \right).$$

1.) Show that

$$h(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

defines a metric on F_X.

2.) Show that for all $A, B, C, D \in F_X$

$$h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}$$

3.) Furthermore show that (F_X, h) is compact if (X, d) is compact.