Homework 4 **Analysis** **Fall 2001**

The problems are due on Tuesday, November 6.

For all students:

Problem 1. (a) Suppose A is a connected subset of the metric space (X, d). Show that B is connected if $A \subseteq B \subseteq cl(A)$.
(b) Show: If B_α is a connected subset of a metric space (X, d) for all $\alpha \in A$, and $B_\alpha \cap B_\beta \neq \emptyset$ for all $\alpha, \beta \in A$, then $\bigcup_{\alpha \in A} B_\alpha$ is connected.

Problem 2. Let (X, d) be a compact metric space, and let $f : X \to X$ be a function satisfying

$$d(f(x), f(y)) < d(x, y) \text{ for all } x \neq y,\ x, y \in X.$$

Show that there is a $z \in X$ with $f(z) = z$.
Hint: Consider $\inf\{d(x, f(x)) \mid x \in X\}$.

Problem 3. (a) Let (X, d) be a countable non-empty metric space. Let $x \in X$. Show that there is a real number $r > 0$, such that

$$\{z \in X \mid d(x, z) = r\} = \emptyset.$$

(b) Let (X, d) be a connected metric space with at least two elements. Show that X is uncountable.

For graduate students:

Problem 1G. The (path-)connected component of an element x in a metric space (X, d) is the union of all (path-)connected subsets of X containing x. Give an example where the connected component and the path-connected component do not coincide. Show that your example works!

Problem 2G. A set A in $\mathbb{R}^n$ is called *nowhere dense*, if the interior of its closure is empty: $int(cl\,A) = \emptyset$. Show that $\mathbb{R}^n$ cannot be written as the countable union of nowhere dense sets.