Homework 5 **Analysis** **Fall 2001**

The problems are due on Tuesday, November 20.

For all students:

Problem 1. Let X and Y be two metric spaces. Show that $f : X \to Y$ is continuous if and only if for all subsets A, B of X with $cl(A) = cl(B)$ in X, their images $f(A)$ and $f(B)$ satisfy $cl(f(A)) = cl(f(B))$ in Y.

Problem 2. Show: If $f : X \to Y$ is uniformly continuous, and (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is a Cauchy sequence in Y.

Is this result also true, if one only assumes that f is continuous?

Problem 3. Let (X, d) be a compact metric space, and $f : X \to X$ be a map satisfying

$$d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X.$$

Denote by

$$f^n := \underbrace{f \circ f \ldots \circ f}_{n \text{ times}}.$$

(a) Show that f is 1-1.
(b) Show that f is onto. Proceed as follows: If $y \notin f(X)$, consider the sequence $(f^n(y))$. Show that this sequence contains no converging subsequence.

For graduate students:

Problem 1+2G. Let $(f_n) : \mathbb{R} \to [0, 1]$ be a sequence of *increasing* functions, i.e. satisfying $f_n(x) \leq f_n(y)$ for all $x \leq y$ and all $n \in \mathbb{N}$. The problem will establish that a subsequence of (f_n) converges pointwise on $\mathbb{R}$.

1. Show that there is a subsequence (f_{n_k}) of (f_n), which converges at all rational points in $\mathbb{R}$, say $\lim_{k\to\infty} f_{n_k}(q) =: f(q)$ for all rational numbers q.
2. For $x \in \mathbb{R}$ define $f(x) := \sup\{f(q) \mid q \leq x \ , \ q \text{ is rational}\}$. Using the monotonicity of the functions involved, show that $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ for all $x \in \mathbb{R}$, at which f is continuous.
3. Show that a further subsequence of (f_{n_k}) converges to f for *all* $x \in \mathbb{R}$.
   Hint: How many points are there, at which f is discontinuous?