Homework 6 **Analysis** **Fall 2001**

The problems are due on Thursday, December 6.

For all students:

Problem 1. For a real number x, define $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.
Let (a_n) be a sequence of real numbers, such that $\sum a_n$ converges, while $\sum |a_n|$ diverges.

1. Show that the partial sums of $\sum a_n^+$ and the partial sums of $\sum a_n^-$ are unbounded increasing sequences. Note that $\lim_{n\to\infty} a_n^+ = 0$ and $\lim_{n\to\infty} a_n^- = 0$.

2. Let $L \in \mathbb{R}$. Using the results above, show that there is a bijection $\rho : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\rho(n)} = L.$$

Problem 2. Let (a_n) be a sequence of real numbers, such that $\sum |a_n|$ converges, and let $\rho : \mathbb{N} \to \mathbb{N}$ be any bijection. Show that

$$\sum_{n=1}^{\infty} a_{\rho(n)} = \sum_{n=1}^{\infty} a_n.$$

Problem 3. Show that the series $1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + \ldots$ converges absolutely for $|x| < 1$. Find the sum of the series.

For graduate students:

Problem 1G. Show that the following limit exists

$$\lim_{n\to\infty} \left\{ \left(\sum_{k=1}^{n} \frac{1}{k} \right) - \ln n \right\}$$

You may use the usual facts about integration and the natural logarithm you learnt in the Calculus sequence.

Problem 2G. Show that $\arctan x$ equals its Taylor series with center $x_0 = 0$ for $|x| < 1$.