Problem 1. Let \( (x_n) \) be a given sequence. Show that the set of subsequences of \( (x_n) \) has the same cardinality as the set of real numbers.

Problem 2. We say that a real number \( x \) is algebraic if \( x \) is a root of a polynomial with integer coefficients.
Thus \( x \) is algebraic if it satisfies \( P(x) = 0 \) for some polynomial
\[
P(z) = \sum_{k=0}^{n} a_k z^k
\]
with \( n \in \mathbb{N} \cup \{0\} \) and \( a_k \in \mathbb{Z} \) for \( k = 0, 1, \ldots, n \). Clearly, all rational numbers are algebraic; \( x = -\sqrt{2} \) is another example of an algebraic number, since it satisfies the polynomial equation \( x^2 - 2 = 0 \).
Show that the set of algebraic numbers is countable.

Problem 3. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\), and define \( g : [a, b] \to \mathbb{R} \) by
\[
g(x) = \sup \{ f(t) \mid a \leq t \leq x \}.
\]
Show that \( g \) is continuous on \([a, b]\).

Problem 4. Let \( a < b \) be real numbers. Suppose the function \( f : (a, b) \to \mathbb{R} \) is uniformly continuous on the open interval \((a, b)\). Then \( f \) can be defined at the endpoints \( a \) and \( b \) in such a way that the extension \( f : [a, b] \to \mathbb{R} \) is continuous on the closed interval \([a, b]\).