Homework 3 **Introduction to Analysis** **Spring 2003**

The problems are due on Monday April 7.

Problem 1. Let (x_n) be a given sequence. Show that the set of subsequences of (x_n) has the same cardinality as the set of real numbers.

Problem 2. We say that a real number x is *algebraic* if x is a root of a polynomial with integer coefficients.

Thus x is algebraic if it satisfies $P(x) = 0$ for some polynomial

$$P(z) = \sum_{k=0}^{n} a_k z^k$$

with $n \in \mathbb{N} \cup \{0\}$ and $a_k \in \mathbb{Z}$ for $k = 0, 1, \ldots, n$. Clearly, all rational numbers are algebraic; $x = -\sqrt{2}$ is another example of an algebraic number, since it satisfies the polynomial equation $x^2 - 2 = 0$.

Show that the set of algebraic numbers is countable.

Problem 3. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = \sup\{f(t) \mid a \leq t \leq x\}.$$

Show that g is continuous on $[a, b]$.

Problem 4. Let $a < b$ be real numbers. Suppose the function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on the open interval (a, b). Then f can be defined at the endpoints a and b in such a way that the extension $f : [a, b] \to \mathbb{R}$ is continuous on the closed interval $[a, b]$.