Theorem: The following four statements are equivalent

(1) Let $S \subseteq \mathbb{N}$. If S satisfies

(a) $1 \in S$

and (b) $n \in S \Rightarrow (n+1) \in S$,

then $S = \mathbb{N}$.

(2) Let $P(n)$ be a predicate with domain $\mathbb{N}$.

Suppose

(a) $P(1)$ is true

and (b) Whenever $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$

(3) Let $P(n)$ be a predicate w/ domain $\mathbb{N}$.

Suppose

(a) $P(1)$ is true

and (b) Whenever $P(k)$ is true for all $k \leq n$, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

(4) Every non-empty subset of $\mathbb{N}$ has a smallest element.

Proof: (1) $\Rightarrow$ (2) Let $S = \{ n \in \mathbb{N} \mid P(n) \text{ is true} \}$

Then by (2a), $1 \in S$. Now suppose $n \in S$, i.e. $P(n)$ is true. Then by (2b), $P(n+1)$ is true, so $(n+1) \in S$. By (1), $S = \mathbb{N}$

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4) Suppose there is a non-empty set K without a smallest element. Let $S = \mathbb{N} \setminus K$.

Let $P(n)$ be the predicate: $n \in S$.

Clearly $P(1)$ is true: if $1 \notin S$, then $1 \in K$, and

thus 1 will be the smallest element in K.
Now suppose $P(k)$ is true for all $k \le n$;
thus $1, 2, 3, \dots, n \in S$. So $1, 2, \dots, n \notin K$.
If $(n+1) \in K$, then $n+1$ is the smallest element of K; thus $(n+1) \notin K$, i.e. $(n+1) \in S$. Thus $P(n+1)$ is true, so $P(n)$ is true for all $n \in \mathbb{N}$. Consequently $S = \mathbb{N}$ and $K = \emptyset$, a contradiction.

(4) $\Rightarrow$ (1) Suppose $S \subseteq \mathbb{N}$ satisfies (1a) and (1b) but $S \subsetneq \mathbb{N}$. Then $K = \mathbb{N} \setminus S \neq \emptyset$.
By (4) it has a smallest element, say $n \in K$ is K's smallest element.
Then $n = 1$ or $n - 1 \notin K$.
In the first case, $1 \notin S$, contradicting (1a). In the second case (1b) applied to $(n-1)$ implies that $n \in S$, so $n \notin K$, contrary to our assumption.

q.e.d.