Math 3325 **Take-Home Exam** **Spring 2005**

You may use our textbook and your own class notes, nothing else. You must not discuss this assignment with anyone. If you have questions, feel free to send me an email at `hknaust@utep.edu`***.***
The solutions are due on Monday, April 4, at the beginning of class.

Problem 1 (10 points) Let $A \subseteq B$. Show that

$$\mathcal{P}(B \setminus A) \setminus \{\emptyset\} \subseteq \mathcal{P}(B) \setminus \mathcal{P}(A).$$

Problem 2 (10 points) For $n \in \mathbb{N}$, let M_n denote the set of integer multiples of n:

$$M_n = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\}.$$

Find

$$\bigcup_{n\in\mathbb{N}} M_n \text{ and } \bigcap_{n\in\mathbb{N}} M_n.$$

Prove that the answers you give are correct.

Problem 3 (10 points) Show that the following equation holds for all $n \in \mathbb{N}$:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 4 (10 points) Recall that a positive integer $p \geq 2$ is called a *prime number* if p is divisible only by 1 and itself. (The eight smallest prime numbers are $2, 3, 5, 7, 11, 13, 17, 19$.)

Show that every natural number $n \geq 2$ can be written as a product of prime numbers: For all $n \in \mathbb{N},\ n \geq 2$ there are $k \in \mathbb{N}$ and prime numbers $p_1, p_2, \ldots, p_k$ such that

$$n = p_1 \cdot p_2 \cdot p_3 \cdots p_k.$$

(Example: $660 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 11$. Note that the product is allowed to have only one factor, example: 17=17.)

Problem 5 (10 points) **(a)** Define a relation $\sim$ on the set of integers $\mathbb{Z}$ as follows:

$$a \sim b \quad \Longleftrightarrow \quad a = \kappa \cdot b \text{ for some } \kappa \in \mathbb{Z}.$$

Check whether this relation is reflexive, symmetric, anti-symmetric, or transitive. (For each property, give a proof or find a counterexample.)

(b) Define a relation $\sim$ on the set of integers $\mathbb{Z}$ as follows:

$$a \sim b \quad \Longleftrightarrow \quad a = p \cdot b \text{ for some prime number } p.$$

Check whether this relation is reflexive, symmetric, anti-symmetric, or transitive. (For each property, give a proof or find a counterexample.)