Math 3325 **Test 1 - Solutions** **Spring 2005**

Problem 1 (15 points) Negate the following statements. Here x is a positive integer.

- x is divisible by 12 and x is divisible by 9.
- If x is divisible by 12, then x is divisible by 9.
- If x is divisible by 5 and x is divisible by 12, then x is divisible by 60 or x is not prime.

Problem 2 (10 points) Negate the following statement: "All dogs have three legs, or there is a cat with two tails."

Negation: "There is a dog which does not have three legs and no cat has two tails."

Problem 3 (15 points) In each case, give an example, or explain why such an example cannot exist:

- Is there a predicate $A(x, y)$ such that the statement

$$\forall x \; \exists y : \; A(x, y)$$

is true, while the statement

$$\exists y \; \forall x : \; A(x, y)$$

is false?

If, e.g. $A(x, y) := \quad x = y$, the first statement is true, while the second one is false.

- Is there a predicate $A(x, y)$ such that the statement

$$\exists y \; \forall x : \; A(x, y)$$

is true, while the statement

$$\forall x \; \exists y : \; A(x, y)$$

is false?

Since the first statement implies the second one (why?), no such example can be found.

Problem 4 (15 points) Use a truth table to show that

$$\neg(A \Rightarrow (B \vee \neg C)) \quad \Longleftrightarrow \quad A \wedge \neg B \wedge C$$

Problem 5 (15 points) Let x be a positive integer. Prove: If x^2 is even, then x is even.

Proof by contraposition: Let x be an odd integer. Then there exists an integer k such that $x = 2k+1$. Note that $2k^2+2k$ is an integer. Consequently $x^2 = 4k^2+4k+1 = 2(2k^2 + 2k) + 1$ is odd.

Problem 6 (15 points) Give the definitions of the following objects:

- The set $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- The set $\mathbb{Q} = \{\frac{p}{q} : \; p, q$ are integers and $q \neq 0\}$
- The interval $(-2, 3] = \{x \in \mathbb{R} : \; -2 < x \leq 3\}$
- The union of two sets: $A \cup B = \{x : \; x \in A$ or $x \in B\}$
- The complement of A in B: $A^{\mathcal{C}}_{B} = \{x \in B : \; x \notin A\}$ (A and B are sets.)

Problem 7 (15 points) Let A and B be two sets. Show: If $(A \cup B) \subseteq (A \cap B)$, then $A = B$.

Proof: Let $x \in A$. Then $x \in A \cup B$. By our hypothesis it follows that $x \in A \cap B$, therefore $x \in B$. $B \subseteq A$ is proved analogously.

"There can never be surprises in logic." (Ludwig Wittgenstein 1889–1951)