1 The Natural Numbers

1.1 Definition

In 1888, RICHARD DEDEKIND suggested a definition of the set of **Natural Numbers** along the following lines[1]:

> *The natural numbers are a set* $\mathbb{N}$ *together with a special element called* 0*, and a function* $S : \mathbb{N} \to \mathbb{N}$ *satisfying the following axioms:*
>
> **(N1)** S *is injective.*[2]
>
> **(N2)** $0 \notin S(\mathbb{N})$.[3]
>
> **(N3)** *If a subset* M *of* $\mathbb{N}$ *contains 0 and satisfies* $S(M) \subseteq M$*, then* $M = \mathbb{N}$.

The function S is called the successor function.

The first two axioms describe the process of counting, the third axiom assures the **Principle of Induction**:

---

[1]Maybe a more famous definition of the natural numbers was introduced by GUISEPPE PEANO in 1889:

> *The natural numbers are a set* $\mathbb{N}$ *together with a special element called* 0*, and a function* $S : \mathbb{N} \to \mathbb{N}$ *satisfying the following axioms:*
>
> **(P1)** $0 \in \mathbb{N}$.
>
> **(P2)** *If* $n \in \mathbb{N}$*, then* $S(n) \in \mathbb{N}$.
>
> **(P3)** *If* $n \in \mathbb{N}$*, then* $S(n) \neq 0$.
>
> **(P4)** *If a set* A *contains* 0*, and if* A *contains* $S(n)$*, whenever it contains* n*, then the set* A *contains* $\mathbb{N}$.
>
> **(P5)** $S(m) = S(n)$ *implies* $m = n$ *for all* $m, n \in \mathbb{N}$.

[2]A function $f : A \to B$ is called *injective* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

[3]For a function $f : A \to B$, $f(A) := \{b \in B \mid f(a) = b \text{ for some } a \in A\}$.

Exercise 1 Let $P(n)$ be a predicate with the set of natural numbers as its domain. If

1. $P(0)$ is true, and
2. $P(S(n))$ is true, whenever $P(n)$ is true,

then $P(n)$ holds for all natural numbers.

1.2 Existence and Uniqueness

Do natural numbers exist? Following Dedekind, we will say that a set M is **infinite**, if there is an injective map $f : M \to M$ that is not surjective.[4]

The existence of natural numbers is then equivalent to the existence of infinite sets:

Task 2 There are infinite sets if and only if there is a set which satisfies Axioms (N1)–(N3).

In order not to get stuck in a finite universe, we will from now on additionally assume that the following axiom holds[5]:

> **(N4)** *There is a set which satisfies Axioms (N1)–(N3).*

Before we give a proof of the "essential" uniqueness of the natural numbers, we will follow Dedekind and establish the following general **Recursion Principle**:

[4] A function $f : A \to B$ is called *surjective*, if $f(A) \neq B$.

[5] JOHN VON NEUMANN introduced the following set-theoretic construction for the set of natural numbers: $0 := \emptyset$ and $S(n) := \{n\}$ for all $n \in \mathbb{N}$. So $1 := \{0\} = \{\emptyset\}, 2 := \{1\} = \{\{\emptyset\}\}, 3 := \{2\} = \{\{\{\emptyset\}\}\}$, etc.

Task 3 Let A be an arbitrary set, and let $a \in A$ and a function $f : A \to A$ be given. Then there exists a unique map $\varphi : \mathbb{N} \to A$ satisfying

1. $\varphi(0) = a$, and
2. $\varphi \circ S = f \circ \varphi$.

The setup of the proof is somewhat tricky: Consider all subsets $K \subseteq \mathbb{N} \times A$ with the following properties:

1. $(0, a) \in K$, and
2. If $(n, b) \in K$, then $(S(n), f(b)) \in K$.

Clearly $\mathbb{N} \times A$ itself has these properties; we can therefore define the smallest such set: Let

$$L = \bigcap \{K \subseteq \mathbb{N} \times A \mid K \text{ satisfies (1) and (2)}\} .$$

Now show by induction that for every $n \in \mathbb{N}$ there is a unique $b \in A$ with $(n, b) \in L$. This property defines φ by setting $\varphi(n) = b$ for all $n \in \mathbb{N}$.

The Recursion Principles makes it possible to define a recursive procedure (the function φ) via a formula (the function f).

The set of natural numbers is unique in the following sense:

Task 4 Suppose that $\mathbb{N}$, $S : \mathbb{N} \to \mathbb{N}$ and 0 satisfy Axioms (N1)–(N3), and that $\mathbb{N}'$, $S' : \mathbb{N}' \to \mathbb{N}'$ and $0'$ satisfy Axioms (N1)–(N3) as well.
Then there is a bijection[6] $\varphi : \mathbb{N} \to \mathbb{N}'$ such that

1. $\varphi(0) = 0'$, and
2. $S' \circ \varphi = \varphi \circ S$.

[6] A function $f : A \to B$ is a *bijection*, if it is both injective and surjective.

1.3 Arithmetic Properties

Addition of natural numbers is established recursively in the following way: For $m, n \in \mathbb{N}$ we define

$$\begin{aligned} m + 0 &:= m \\ m + S(n) &:= S(m + n) \end{aligned}$$

Task 5 Use the Recursion Principle to make this procedure precise.

Note that we now know in particular that for all natural numbers $S(n) = n + 1$ (here $S(0) := 1$.)

Task 6 Show that addition on $\mathbb{N}$ is associative.

Task 7 Show that addition on $\mathbb{N}$ is commutative.

This last exercise implies in particular that 0 is the (unique) neutral element with respect to addition: $n + 0 = 0 + n$ holds for all $n \in \mathbb{N}$.

Task 8 Show that the following cancellation law holds for natural numbers: If $k + m = l + m$, then $k = l$.

Multiplication of natural numbers is also defined recursively as follows: For $m, n \in \mathbb{N}$ we define

$$\begin{aligned} m \cdot 0 &:= 0 \\ m \cdot (n + 1) &:= m \cdot n + m \end{aligned}$$

Task 9 Show that the following distributive law holds for natural numbers: $(m+n) \cdot k = m \cdot k + n \cdot k$.

Task 10

1. Show that multiplication on $\mathbb{N}$ is commutative.
2. Show that multiplication on $\mathbb{N}$ is associative.
3. Show that 1 is the neutral element with respect to multiplication.
4. Show that the following cancellation law holds for natural numbers: Suppose $m \neq 0$. If $m \cdot k = m \cdot l$, then $k = l$.

Finally we can impose a **total order**[7] on $\mathbb{N}$ as follows: We say that $m \leq n$, if there is a natural number k, such that $m + k = n$.

Task 11 Show that "$\leq$" is indeed a total order:

1. "$\leq$" is reflexive.[8]
2. "$\leq$" is anti-symmetric.[9]
3. "$\leq$" is transitive.[10]
4. For all $m, n \in \mathbb{N}$, $m \leq n$ or $n \leq m$.

[7] A relation $\sim$ on A is called a *total order*, if $\sim$ is reflexive, anti-symmetric, transitive, and has the property that for all $a, b \in A$, $a \sim b$ or $b \sim a$ holds.

[8] A relation $\sim$ on A is *reflexive* if for all $a \in A$, $a \sim a$.

[9] A relation $\sim$ on A is *anti-symmetric* if for all $a, b \in A$ the following holds: $a \sim b$ and $b \sim a$ implies that a=b.

[10] A relation $\sim$ on A is *transitive* if for all $a, b, c \in A$ the following holds: $a \sim b$ and $b \sim c$ implies that $a \sim c$.

Task 12 Show the following compatibility laws:

1. If $m \leq n$, then $m + k \leq n + k$ for all $k \in \mathbb{N}$.
2. If $m \leq n$, then $m \cdot k \leq n \cdot k$ for all $k \in \mathbb{N}$.

2 The Integers

2.1 Definition

Integers can be written as differences of natural numbers. The set of integers $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ will therefore be defined as certain equivalence classes of the two-fold Cartesian product of $\mathbb{N}$.

We define a relation on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c.$$

Exercise 13 Show that "$\sim$" defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$:

1. "$\sim$" is reflexive.
2. "$\sim$" is symmetric.[11]
3. "$\sim$" is transitive.

We will denote equivalence classes as follows:

$$(a, b)_\sim := \{(c, d) \mid (c, d) \sim (a, b)\}.$$

[11] A relation $\sim$ on A is called *symmetric*, if for all $a, b \in A$ the following holds: $a \sim b$ implies $b \sim a$.

The set of integers $\mathbb{Z}$ is the set of all equivalence classes thus obtained:

$$\mathbb{Z} = \{(a, b)_\sim \mid a, b \in \mathbb{N}\}.$$

2.2 Arithmetic Properties

Addition of integers will be defined component-wise:

$$(a, b)_\sim + (c, d)_\sim = (a + c, b + d)_\sim.$$

The next two exercises will show that $\mathbb{Z}$ is an *Abelian group*[12] with respect to addition.

Exercise 14
1. Show that the addition of integers is well-defined (i.e. independent of the chosen representatives of the equivalence classes).
2. Show that the addition of integers is commutative.
3. Show that the addition of integers is associative.

Exercise 15
1. Show that the addition of integers has $(0, 0)_\sim$ as its neutral element.
2. Show that for all $a, b \in \mathbb{N}$ the following holds: $(a, b)_\sim + (b, a)_\sim = (0, 0)_\sim$. Thus every element in $\mathbb{Z}$ has an inverse element.

Exercise 16
1. The map $\phi : \mathbb{N} \to \mathbb{Z}$ defined by $\phi(n) = (n, 0)_\sim$ is injective.

[12] A set G with a binary operation $\star$ is called an *Abelian group* if $\star$ is commutative and associative, if $(A, \star)$ has a neutral element n satisfying $g \star n = g$ for all $g \in G$, and if $(A, \star)$ has inverse elements, i.e., for all $g \in G$ there is an $h \in G$ satisfying $g \star h = n$.

2. For all $m, n \in \mathbb{N}$ the following holds: $\phi(m) + \phi(n) = \phi(m + n)$.

From now on we will identify $\mathbb{N}$ with $\phi(\mathbb{N})$ and write $a - b$ instead of $(a, b)_\sim$. For instance -5 is the equivalence class of all elements equivalent to $(0, 5)$.

I will leave it to the reader to define integer **multiplication**:

Task 17 Define integer multiplication (make sure it is well-defined), and prove some of the familiar properties:

1. $(\mathbb{Z}, \cdot)$ is commutative, associative, and has 1 as its neutral element.
2. $(\mathbb{Z}, \cdot)$ is zero divisor-free: $m \cdot n = 0$ implies $m = 0$ or $n = 0$.
3. $(m + n) \cdot k = m \cdot k + n \cdot k$ for all $m, n, k \in \mathbb{Z}$.

Last not least we will define a **total order** on $\mathbb{Z}$ as follows:

$$m \leq n \text{ if and only if } n - m \in \mathbb{N}.$$

Exercise 18 1. Show that "$\leq$" defines a total order on $\mathbb{Z}$.

2. If $m \leq n$, then $m + k \leq n + k$ for all $k \in \mathbb{Z}$.
3. If $m \leq n$ and $0 \leq k$, then $m \cdot k \leq n \cdot k$.

3 The Rational Numbers

Once again we define the next larger set as certain equivalence classes. On $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$, we define a relation $\cong$ as follows:

$$(a, b) \cong (c, d) \text{ if and only if } a \cdot d = b \cdot c.$$

We write equivalence classes in the familiar way

$$\frac{a}{b} = \{(c,d) \mid (c,d) \cong (a,b)\},$$

and denote the rational numbers by

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus \{0\} \right\}.$$

For integers n we write n instead of $\frac{n}{1}$.

With the natural addition and multiplication

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

and the order induced by

$$0 \le \frac{a}{b} \text{ if and only if } (0 \le a \text{ and } 0 \le b) \text{ or } (a \le 0 \text{ and } b \le 0),$$

the set of rational numbers becomes an **ordered field**:

Theorem. $(\mathbb{Q}, +, \cdot, \le)$ has the following properties:

1. $(\mathbb{Q}, +)$ is an Abelian group with neutral element 0.
2. $(\mathbb{Q} \setminus \{0\}, \cdot)$ is an Abelian group with neutral element 1.
3. $(\mathbb{Q}, \le)$ is a total order.
4. $(a+b) \cdot c = a \cdot c + b \cdot c$.
5. (a) If $a + c = b + c$, then $a = b$.
   (b) If $c \neq 0$ and $a \cdot c = b \cdot c$, then $a = b$.
   (c) If $a \cdot b = 0$, then $a = 0$ or $b = 0$.
6. (a) $a \le b$ implies $a + c \le b + c$ for all $a, b, c \in \mathbb{Q}$.
   (b) $a \le b$ implies $a \cdot c \le b \cdot c$ for all $a, b, c \in \mathbb{Q}$ with $0 \le c$.

The rational numbers have two more interesting properties. Let us write $a < b$ if $a \leq b$ and $a \neq b$. We will say that a is *positive*, if $0 < a$. Similarly, a is called *negative*, if $0 < -a$.

Exercise 19 $\mathbb{Q}$ is *dense in itself*: For all $a, b \in \mathbb{Q}$ with $a < b$ there is a $c \in \mathbb{Q}$ with $a < c < b$.

Exercise 20 $\mathbb{Q}$ is *Archimedean*: For all positive $a, b \in \mathbb{Q}$, there is a natural number n such that $b < n \cdot a$.

4 The Real Numbers

4.1 Completeness

While the rational numbers have nice algebraic properties with respect to their addition, their multiplication and their order, they have one crucial deficiency: They have "holes".

For instance, the increasing sequence of rational numbers

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

approaches the non-rational number $\sqrt{2}$, a fact well known since antiquity.

We want to remedy this deficiency: we want to construct an ordered field F containing the rational numbers, which is "complete" in the following sense:

> **(C1)** Every increasing[13] bounded[14] sequence[15] of elements in F converges[16] to an element in F.

It is convenient to describe completeness also in a slightly different way.

We say a non-empty set $A \subseteq F$ is *bounded from above*, if there is a $b \in F$ such that $a \leq b$ for all $a \in A$.

If $A \subseteq F$ is bounded from above, we say that A has a *least upper bound*, denoted by $\sup(A) \in F$, if

1. $\sup(A)$ is an upper bound of A, and
2. for all upper bounds b of A, we have $\sup(A) \leq b$.

Note that we do not require that $\sup(A)$ is an element of A!

[13] A sequence $\phi : \mathbb{N} \to F$ is called *increasing*, if $m \leq n$ implies $\phi(m) \leq \phi(n)$.

[14] A sequence $\phi : \mathbb{N} \to F$ is called *bounded*, if there is a $b \in F$ such that $\phi(n) \leq b$ for all $n \in \mathbb{N}$.

[15] A *sequence* is a function $\phi : \mathbb{N} \to F$.

[16] We say that the increasing sequence ϕ *converges* to $a \in F$, if for all $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $|\phi(n) - a| \leq \varepsilon$.

Exercise 21 Let $A = \{a \in \mathbb{Q} \mid a^2 < 2\}$; then A is bounded form above, but fails to have a least upper bound in $\mathbb{Q}$.

We can also state completeness as follows:

(C2) Every subset A of F, which is bounded from above, has a least upper bound.

Task 22 Show that both axioms (C1) and (C2) are equivalent.

Historically, two "constructions" of the real numbers gained prominence in the 19th century, due to AUGUSTIN-LOUIS CAUCHY, and to RICHARD DEDEKIND, respectively. We present both constructions below.

4.2 Cauchy Sequences

One construction of the set of real numbers is due to Cauchy: We say that a sequence $\phi : \mathbb{N} \to \mathbb{Q}$ of rational numbers is a *Cauchy sequence*, if it satisfies the following property: For all rational $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that $|\phi(m) - \phi(n)| < \varepsilon$ for all natural numbers $m, n \geq N$.

Exercise 23 Every Cauchy sequence is bounded[17].

We set
$$\mathcal{C} = \{\phi : \mathbb{N} \to \mathbb{Q} \mid \phi \text{ is a Cauchy sequence}\}.$$

[17]We say a sequence $\phi : \mathbb{N} \to \mathbb{Q}$ is *bounded*, if there are rational numbers a and b such that $a \leq \phi(n) \leq b$ for all $n \in \mathbb{N}$.

On $\mathcal{C}$ we define a relation $\simeq$ as follows: $\phi \simeq \psi$ if and only if for all rational $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|\phi(n) - \psi(n)| < \varepsilon$ for all natural numbers $n \geq N$.

We can't quite say it like that, but what this "really" means is that two sequences are equivalent if they converge to the same "limit".

It is trivial to observe that $\simeq$ is reflexive and symmetric, and it is easy to show that $\simeq$ is transitive:

Exercise 24 The relation $\simeq$ is transitive.

Cauchy then defined the set of real numbers $\mathbb{R}$ as equivalence classes of elements of $\mathcal{C}$ under the relation $\simeq$:

$$\phi_{\simeq} := \{\psi : \mathbb{N} \to \mathbb{Q} \mid \psi \simeq \phi\}.$$

$$\mathbb{R} := \{\phi_{\simeq} \mid \phi \in \mathcal{C}\}.$$

For instance, if ϕ is the Cauchy sequence

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots,$$

then $\phi_{\simeq}$ is the equivalence class of all sequences of rational numbers that converge to the real number $\sqrt{2}$, so $\phi_{\simeq}$ "=" $\sqrt{2}$.

The equivalence class of a constant rational sequence defines that constant.

Task 25 Both addition and multiplication of real numbers are defined coordinate-wise:

$$\phi_{\simeq} + \psi_{\simeq} := (\phi + \psi)_{\simeq},$$

where $(\phi + \psi)(n) := \phi(n) + \psi(n)$ for all $n \in \mathbb{N}$.

Similarly,

$$\phi_{\simeq} \cdot \psi_{\simeq} := (\phi \cdot \psi)_{\simeq},$$

where $(\phi \cdot \psi)(n) := \phi(n) \cdot \psi(n)$ for all $n \in \mathbb{N}$.

Show that addition and multiplication of real numbers are well-defined.

It is then straightforward to show that the real numbers with this addition and multiplication form a field.

It is more complicated to define an order on $\mathbb{R}$: We say that $\phi_{\simeq} \leq \psi_{\simeq}$ if for all rational $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\phi(n) \leq \psi(n) + \varepsilon$ for all $n \geq N$.

Task 26 Show that "$\leq$" is well-defined.

Task 27 Show that "$\leq$" defines a total order on $\mathbb{R}$.

Indeed, the real numbers are an **ordered Archimedean field**:

Theorem. $(\mathbb{R}, +, \cdot, \leq)$ has the following properties:

1. $(\mathbb{R}, +)$ is an Abelian group with neutral element 0.
2. $(\mathbb{R} \setminus \{0\}, \cdot)$ is an Abelian group with neutral element 1.
3. $(\mathbb{R}, \leq)$ is a total order.
4. $(a + b) \cdot c = a \cdot c + b \cdot c$.
5. (a) If $a + c = b + c$, then $a = b$.
   (b) If $c \neq 0$ and $a \cdot c = b \cdot c$, then $a = b$.

(c) If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

6. (a) $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in \mathbb{R}$.

   (b) $a \leq b$ implies $a \cdot c \leq b \cdot c$ for all $a, b, c \in \mathbb{R}$ with $0 \leq c$.

7. For all positive $a, b \in \mathbb{R}$, there is a natural number n such that $b < n \cdot a$.

Last not least, the real numbers $\mathbb{R}$ are **complete**:

Task 28 Show that $\mathbb{R}$ satisfies Axiom (C1).

To prove this, start with an increasing sequence $\Phi : \mathbb{N} \to \mathbb{R}$. To say that the sequence is bounded means that there is a real number $\psi_{\simeq}$ such that $(\Phi_{\simeq})(n) \leq \psi_{\simeq}$ for all $n \in \mathbb{N}$. The tricky part is to come up with a candidate for a sequence $\sigma : \mathbb{N} \to \mathbb{Q}$ of rational numbers so that $\sigma_{\simeq}$ is the limit of the sequence $\Phi : \mathbb{N} \to \mathbb{R}$.

Task 29 Show that the set of rational numbers is dense in the set of real numbers. This means that for every two real numbers $x < y$, there is a rational number q satisfying $x < q < y$.

4.3 Dedekind Cuts

Dedekind gives an alternative construction of the real numbers as follows: Given two sets of rational numbers $\emptyset \neq L, U \subseteq \mathbb{Q}$, we say that (L, U) is a *partition* of $\mathbb{Q}$ (into two sets), if $L \cup U = \mathbb{Q}$ and $L \cap U = \emptyset$.

A partition (L, U) of $\mathbb{Q}$ is called a *Dedekind cut*, if the following properties hold:

1. If $a \in L$ and $b \in U$, then $a < b$.

2. L has no maximal element.[18]

It is really superfluous to list both L and U, since L and U are complementary sets: $U = \mathbb{Q} \setminus L$, and $L = \mathbb{Q} \setminus U$.

Here is a "boring" Dedekind cut:

$$L = \{q \in \mathbb{Q} \mid q < -3\}, \; U = \{q \in \mathbb{Q} \mid q \geq -3\}.$$

Task 30 Show that

$$L = \{q \in \mathbb{Q} \mid q \leq 0 \text{ or } q^2 < 2\}, \; U = \{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 > 2\}$$

defines a Dedekind cut.

Note that the "gap" in the cut is the real number $\sqrt{2}$.

Dedekind then defined the set of real numbers to be the set of all Dedekind cuts:

$$\mathbb{R} = \{(L, U) \mid (L, U) \text{ is a Dedekind cut}\}.$$

Note that a rational number $q \in \mathbb{Q}$ corresponds to the Dedekind cut, defined by $L = (-\infty, q) \cap \mathbb{Q}$, $U = [q, \infty) \cap \mathbb{Q}$.

Given two Dedekind cuts (L_1, U_1) and (L_2, U_2) we define their sum to be the Dedekind cut (X, Y), where

$$X = \{x \in \mathbb{Q} \mid x = l_1 + l_2 \text{ for some } l_1 \in L_1 \text{ and } l_2 \in L_2\},$$

and

$$Y = \{y \in \mathbb{Q} \mid y = u_1 + u_2 \text{ for some } u_1 \in U_1 \text{ and } u_2 \in U_2\}.$$

[18] The element x of a set A of rational numbers is called *maximal element* of A, if $x \geq a$ for all $a \in A$.

Exercise 31 Show that (X, Y) is indeed a Dedekind cut.

Task 32 Show that the Dedekind cuts with the addition defined above form an Abelian group. What is the neutral element? What is the additive inverse of a Dedekind cut?

Note that the previous task makes it possible to define the difference of two Dedekind cuts.

Next, we can define an order on Dedekind cuts: We say that $(L_1, U_1) \leq (L_2, U_2)$, if $L_1 \subseteq L_2$. In particular, (L, U) is non-negative, if $(-\infty, 0) \cap \mathbb{Q} \subseteq L$.

It is harder to define the multiplication of Dedekind cuts. If both (L_1, U_1) and (L_2, U_2) are non-negative, we define their product (X, Y) as

$$X = \{x \in \mathbb{Q} \mid x = l_1 \cdot l_2 \text{ for some } l_1 \in L_1 \text{ and } l_2 \in L_2\},$$

and

$$Y = \{y \in \mathbb{Q} \mid y = u_1 \cdot u_2 \text{ for some } u_1 \in U_1 \text{ and } u_2 \in U_2\}.$$

Exercise 33 Check that the product defined above is indeed a Dedekind cut.

To define the product of other Dedekind cuts, we need the following result:

Task 34 Every Dedekind cut is the difference of two non-negative Dedekind cuts.

We then define the product of two arbitrary Dedekind cuts by “multiplying out”.

Task 35 Define the product of two arbitrary Dedekind cuts formally, and show that the concept is well-defined.

With these definitions one can show

Theorem. The real numbers with the addition, multiplication and order defined above form an **ordered Archimedean field**.

The most interesting part of the Theorem is contained in the task below:

Task 36 Show the existence of a multiplicative neutral Dedekind cut, and the existence of multiplicative inverse Dedekind cuts.

Most importantly, the set of real numbers defined via Dedekind cuts is **complete**:

Task 37 Show that the set of all Dedekind cuts satisfies Axiom (C2).