A Remark for Task 28

You need to know what it means for a sequence of real numbers to converge.

Given a sequence $\Phi : \mathbb{N} \to \mathbb{R}$ of real numbers and a real number σ, we say that $(\Phi(n))$ CONVERGES TO σ, if for all rational $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$-\varepsilon \leq \Phi(n) - \sigma \leq \varepsilon$$

for all $n \geq N$.

What exactly does this mean? Here we identify a rational number ε with its canonical image in $\mathbb{R}$ (the constant sequence $\varepsilon : \mathbb{N} \to \mathbb{Q}$, where $\varepsilon(n) = \varepsilon$ for all $n \in \mathbb{N}$); we use the order "$\leq$" for real numbers defined earlier.

One can similarly define what it means for a sequence of real numbers to be a Cauchy sequence: Given a sequence $\Phi : \mathbb{N} \to \mathbb{R}$ of real numbers, we say that $(\Phi(n))$ is a CAUCHY SEQUENCE, if for all rational $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$-\varepsilon \leq \Phi(n) - \Phi(m) \leq \varepsilon$$

for all $n, m \geq N$. One can then prove the following result:

Theorem. Every Cauchy sequence of real numbers converges to a real number.