Problem: *Let $\underline{a} < \overline{a}$ and $\underline{b} < \overline{b}$ be real numbers, and let R denote the rectangle*

$$R = \{(x, y) \in \mathbb{R}^2 \mid \underline{a} \leq x \leq \overline{a},\ \underline{b} \leq y \leq \overline{b}\}.$$

Denote by $\mathcal{H}$ the collection of all rectangles S such that (1) S is contained in R, (2) S has sides parallel to the coordinate axes, and (3) S has positive area.

Show that every non-empty subset $\mathcal{K}$ of $\mathcal{H}$ has a least upper bound.

Proof. Let $\mathcal{K} \subseteq \mathcal{H}$, $\mathcal{K} \neq \emptyset$ be given. For a suitable index set Λ we can describe $\mathcal{K}$ as $\mathcal{K} = \{S_\lambda \mid \lambda \in \Lambda\}$, where each rectangle S_λ is of the form

$$S_\lambda = \{(x, y) \in \mathbb{R}^2 \mid \underline{x}_\lambda \leq x \leq \overline{x}_\lambda,\ \underline{y}_\lambda \leq y \leq \overline{y}_\lambda\}$$

for suitable real numbers $\underline{a} \leq \underline{x}_\lambda < \overline{x}_\lambda \leq \overline{a}$, and $\underline{b} \leq \underline{y}_\lambda < \overline{y}_\lambda \leq \overline{b}$.

We define

$$S = \{(x, y) \in \mathbb{R}^2 \mid \underline{x} \leq x \leq \overline{x},\ \underline{y} \leq y \leq \overline{y}\},$$

where $\underline{x} = \mathrm{glb}\{\underline{x}_\lambda \mid \lambda \in \Lambda\}$, $\overline{x} = \mathrm{lub}\{\overline{x}_\lambda \mid \lambda \in \Lambda\}$, $\underline{y} = \mathrm{glb}\{\underline{y}_\lambda \mid \lambda \in \Lambda\}$, and $\overline{y} = \mathrm{lub}\{\overline{y}_\lambda \mid \lambda \in \Lambda\}$.

We claim that S is the least upper bound of $\mathcal{K}$. Since $\underline{x} \leq \underline{x}_\lambda \leq \overline{x}_\lambda \leq \overline{x}$ and $\underline{y} \leq \underline{y}_\lambda \leq \overline{y}_\lambda \leq \overline{y}$ for all $\lambda \in \Lambda$, S is indeed an upper bound for $\mathcal{K}$. Now suppose

$$S' = \{(x, y) \in \mathbb{R}^2 \mid \underline{x}' \leq x \leq \overline{x}',\ \underline{y}' \leq y \leq \overline{y}'\}$$

is another upper bound for $\mathcal{K}$. Then $S_\lambda \subseteq S'$ for all $\lambda \in \Lambda$, and therefore, using the definition of $\underline{x}$, $\overline{x}$, $\underline{y}$, and $\overline{y}$ as least upper bounds and greatest lower bounds, respectively, it follows that $\underline{x}' \leq \underline{x} \leq \overline{x} \leq \overline{x}'$ and $\underline{y}' \leq \underline{y} \leq \overline{y} \leq \overline{y}'$, implying that $S \subseteq S'$.