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ON c_0 SEQUENCES IN BANACH SPACES

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ABSTRACT

A Banach space has property (S) if every normalized weakly null sequence contains a subsequence equivalent to the unit vector basis of c_0. We show that the equivalence constant can be chosen "uniformly", i.e., independent of the choice of the normalized weakly null sequence. Furthermore we show that a Banach space with property (S) has property (u). This solves in the negative the conjecture that a separable Banach space with property (u) not containing l_1 has a separable dual.

1. Introduction

A Banach space X is said to have *property* (S) if every normalized weakly null sequence in X admits a subsequence which is C-equivalent to the unit vector basis of c_0 for some $C < \infty$. If the constant C is independent of the particular sequence we say X has *uniform* (S) or (US). A second property relating the internal structure of a Banach space to that of c_0 is property (u). One way of formulating this property is to say X has *property* (u) if whenever (x_n) is a weak Cauchy but not weakly convergent sequence in X, there exists (y_n), a block basis of convex combinations of (x_n), which is equivalent to the summing basis for c_0.

The definition of property (u) is due to A. Pełczyński [P]. He defined the property as follows. If $x^{**} \in X^{**}$ is the w^*-limit of a sequence in X then there exists $(y_n) \subseteq X$, which converges w^* to x^{**} and satisfies

$$\sum_{n=1}^{\infty} |x^*(y_{n+1}) - x^*(y_n)| < \infty \quad \text{for all } x^* \in X^*.$$

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(In the terminology of [HOR], $B_1(X) \subseteq DBSC(X)$.) The equivalence of our definition and Pełczyński's was noted in [HOR] and follows easily from the fact that if $(x_n) \subseteq X$ also converges ω^* to x^{**}, then $\operatorname{dist}(\operatorname{conv}(x_n), \operatorname{conv}(y_n)) = 0$. By [BP1] and [R], if X has property (u) and Y is any infinite dimensional subspace of X, then Y is reflexive or contains c_0 or l_1. Since every subspace of a space with unconditional basis has property (u) [P], it was conjectured by J. Hagler that if X is a separable space with property (u) and not containing l_1, then X^* is separable (see [H]).

In §2 we prove that property (S) implies property (u). In view of the tree space JH constructed by Hagler [H] this yields a negative answer to the conjecture. Indeed Hagler showed JH has property (S), does not contain l_1 and has nonseparable dual.

Property (S) was considered by P. Cembranos in [C]. It was noted to be equivalent to the "hereditary Dunford Pettis property": every (infinite dimensional) subspace of X has the Dunford Pettis property. This equivalence follows easily from the deep "nearly unconditional" theorem (Theorem 2.4 below) of J. Elton ([E]; see also [O]). The question whether (S) implies (US) is raised in [C] (and was originally brought to our attention by A. Pełczyński). We show this to be true in §3. Part of our argument requires a generalization of Elton's argument for the aforementioned theorem.

A corollary of our two main results (see Corollary 2.3) is that X has property (S) iff there exists $C < \infty$ so that whenever $(x_n) \subseteq Ba(X)$ is weak Cauchy, there exists a subsequence (x'_n) with

$$\sum_{n=1}^{\infty} |x^*(x'_{n+1}) - x^*(x'_n)| \leqq C \qquad \text{for all } x^* \in Ba(X^*).$$

(Equivalently

$$\left\| \sum_{n=1}^{k} \varepsilon_n (x'_{n+1} - x'_n) \right\| \leqq C \qquad \text{for all } k \text{ and } \varepsilon_i = \pm 1.\Big)$$

This contrasts nicely with property (u) which may be described similarly except that (x'_n) is not necessarily a subsequence of (x_n) but rather a block basis of convex combinations of (x_n).

We use standard Banach space terminology as may be found in the books [LT] or [D]. The proofs of both our main results require some Ramsey theory (as can be found in [O], [LT] or [D]). The *summing basis for* c_0 is the basis (s_n) given by $s_n = \Sigma_{i=1}^n e_i$, where (e_i) is the unit vector basis of c_0. Finally, it is perhaps worth noting that l_1 has property (S) and by [R], if X has property (S), then every infinite dimensional subspace of X contains l_1 or c_0. Both properties (S) and (u) are hereditary (the later case is due to Pełczyński [P]).

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2. Property (S) implies Property (u)

THEOREM 2.1. *If X has property (S), then X has property (u).*

We first review the Ramsey theorem we require. If M is an infinite subsequence of N, $[M]$ denotes the set of all (infinite) subsequences of M. τ is the pointwise topology on [N], i.e., the relative topology of $[\mathrm{N}] \subseteq 2^{\mathrm{N}}$, given the product topology. $\mathscr{A} \subseteq [\mathrm{N}]$ is said to be *Ramsey* if for all $M \in [\mathrm{N}]$ there exists $L \in [M]$ such that either $[L] \subseteq \mathscr{A}$ or $[L] \subseteq [\mathrm{N}] \setminus \mathscr{A}$. It is known that if $\mathscr{A}$ is τ-Borel then $\mathscr{A}$ is Ramsey [GP]. For a proof of this result, some history and more general results see [O].

PROOF OF THEOREM 2.1. Let $(x_n) \subseteq Ba(X)$ be weak Cauchy but not weakly convergent. By passing to a subsequence we may assume that (x_n) is basic and moreover (y_n) is seminormalized basic where $y_n \equiv x_{n+1} - x_n$ [BP1]. (Since X has property (S), we could have also assumed, by passing to a subsequence, that (y_{2n}) or (y_{2n-1}) is equivalent to the unit vector basis of c_0. If we could obtain this simultaneously for both sequences, we would be finished and this is where Ramsey theory enters.)

For k and $K \in \mathrm{N}$ define

$$\mathscr{A}_k(K) = \left\{ M \in [\mathrm{N}] : M = (m_i) \text{ satisfies } \left\| \sum_{i=1}^{k} \varepsilon_i (x_{m_{2i}} - x_{m_{2i-1}}) \right\| \leqq K \right.$$

$$\left. \text{for all } \varepsilon = \pm 1,\ 1 \leqq i \leqq k \right\}.$$

$\mathscr{A}_k(K)$ is τ-closed and thus $\mathscr{A}(K) \equiv \bigcap_{k=1}^{\infty} \mathscr{A}_k(K)$ is also τ-closed and $\mathscr{A} \equiv \bigcup_{k=1}^{\infty} \mathscr{A}(K)$ is τ-Borel. Consequently $\mathscr{A}$ is Ramsey. Choose $M = (m_i) \in [\mathrm{N}]$ so that either $[M] \subseteq \mathscr{A}$ or $[M] \subseteq [\mathrm{N}] \setminus \mathscr{A}$. Since X has property (S) we obtain $[M] \subseteq \mathscr{A}$. Thus $M \in \mathscr{A}(K_1)$ and $(m_i)_{i=2}^{\infty} \in \mathscr{A}(K_2)$ for some K_1, K_2. It follows that for $x^* \in Ba(X^*)$,

$$\sum_{i=1}^{\infty} |x^*(x_{m_{i+1}}) - x^*(x_{m_i})|$$

$$= \sum_{i=1}^{\infty} |x^*(x_{m_{2i}}) - x^*(x_{m_{2i-1}})| + \sum_{i=1}^{\infty} |x^*(x_{m_{2i+1}}) - x^*(x_{m_{2i}})|$$

$$\leqq K_1 + K_2.$$

In Pełczyński's terminology [P], $\Sigma\, x_{m_i}$ is a w.u. C. In particular $(x_{m_{i+1}} - x_{m_i})_{i=1}^{\infty}$ is equivalent to the unit vector basis of c_0 and so (x_{m_i}) is equivalent to the summing basis for c_0. ■

REMARK 2.2. If X has property (US), the proof yields a fixed K satisfying: if (x_n) is a weak Cauchy sequence in $Ba(X)$ then there exists a subsequence (x_{m_i}) with

$$\sum_{i=1}^{\infty} |x^*(x_{m_{i+1}}) - x^*(x_{m_i})| \leqq K \qquad \text{for all } x^* \in Ba(X^*). \tag{2.1}$$

In fact this turns out to be an equivalence.

COROLLARY 2.3. *X has property (US) iff there exists $K < \infty$ such that if $(x_n) \subseteq Ba(X)$ is weak Cauchy, then there exists a subsequence (x_{m_i}) of (x_n) satisfying* (2.1).

The proof requires Elton's nearly unconditional theorem which we first recall.

THEOREM 2.4 (Elton [E]). *For $0 < \delta \leqq 1$ there exists a constant $K(\delta) < \infty$ such that if (x_n) is a normalized weakly null sequence in a Banach space, then there exists a basic subsequence (x_n') with the following property. If $(a_i)_1^n \subseteq \mathbf{R}$ with $|a_i| \leqq 1$ for all i, and $F \subseteq \{i : |a_i| \geqq \delta\}$, then*

$$\left\| \sum_{i \in F} a_i x_i' \right\| \leqq K(\delta) \left\| \sum_{i=1}^{\infty} a_i x_i' \right\| . \tag{2.2}$$

PROOF OF COROLLARY 2.3. By Remark 2.2 it suffices to show that X has property (US) if it satisfies the condition in the corollary. Let (x_n) be a normalized weakly null sequence in X which satisfies both the conclusion of Theorem 2.4 and condition (2.1). We may assume that $2^{-1} \sup |a_i| \leqq \| \Sigma\, a_i x_i \|$ and thus we have (x_{2n}) is $2 \cdot K \cdot K(1)$-equivalent to the unit vector basis of c_0. Indeed if $F \subseteq \mathbf{N}$ is finite, then by (2.2) and (2.1)

$$\left\| \sum_{n\in F} \pm x_{2n} \right\| \leqq K(1) \left\| \sum_{n\in F} \pm (x_{2n} - x_{2n-1}) \leqq K \cdot K(1). \right. \qquad \blacksquare$$

3. Property (S) implies Property (US)

THEOREM 3.1. *If X has property (S) then X has property (US).*

One's first thoughts on this theorem are that it is false. The counterexample should be $X = (\Sigma X_n)_{c_0}$ where the X_n's are a sequence of bad c_0's (e.g., $X_n = C(\omega^n)$). However it is easy to construct in such a space a normalized weakly null sequence without a c_0-subsequence. Theorem 3.1 is proved by showing that this construction can be carried out in general. We give some definitions to make this precise.

A sequence (x_n) in a Banach space X is called a *c_0-sequence* if $\| x_i \| \leqq 1$ for all i and (x_i) is equivalent to the unit vector basis of c_0. For $M < \infty$ we say (x_i) is an *M-bad c_0-sequence* if (x_i) *is a* c_0-sequence with the additional property that for all subsequences (x_i') of (x_i) there exists $k \in \mathbf{N}$ such that $\| \Sigma_{i=1}^k x_i' \| > M$. The following proposition, due to W. B. Johnson (see [O]), yields that if X has property (S) but fails to have (US), then X contains M-bad c_0-sequences for all M.

PROPOSITION 3.2. *Let (x_i) be a c_0-sequence and let $M < \infty$. Then there exists a subsequence (x_i') of (x_i) such that either*

(a) *(x_i') is an M-bad c_0-sequence, or*

(b) $\| \Sigma_{i\in F} x_i' \| \leqq M$ for all finite $F \subseteq \mathbf{N}$.

PROOF. Let

$$\mathscr{A} = \left\{ L = (l_j) \in [\mathbf{N}] : \left\| \sum_{j=1}^{k} x_{l_j} \right\| \leqq M \text{ for all } k \in \mathbf{N} \right\}.$$

$\mathscr{A}$ is τ-closed and therefore Ramsey. Choose $L \in [\mathbf{N}]$ such that either $[L] \subseteq [\mathbf{N}] \setminus \mathscr{A}$ or $[L] \subseteq \mathscr{A}$ and let $(x_i') = (x_i)_{i\in L}$. In the first case we obtain (a) and in the second (b) holds. $\blacksquare$

We continue with some more definitions. A collection $(x_i^n)_{i,n\in\mathbf{N}} \subseteq X$ is called an *array* in X. An array (y_i^n) is a *subarray* of the array (x_i^n) if there exists $(m_n)_{n=1}^\infty \in [\mathbf{N}]$ such that for all $n \in \mathbf{N}$, $(y_i^n)_{i=1}^\infty$ is a subsequence of $(x_i^{m_n})_{i=1}^\infty$. An array (x_i^n) is a *bad c_0-array* if there exists $M_n \to \infty$ such that for all $n \in \mathbf{N}$, $(x_i^n)_{i=1}^\infty$ is an M_n-bad c_0-sequence.

A bad c_0-array (x_i^n) satisfies the *array procedure* (ARP) if

$$
\text{(ARP)} \quad \begin{cases} \text{there exists a subarray } (y_i^n) \text{ of } (x_i^n) \text{ and reals } a_n > 0 \text{ with } \sum_{n=1}^{\infty} a_n \leqq 1 \\ \text{such that if } y_i = \sum_{n=1}^{\infty} a_n y_i^n, \text{ then } (y_i) \text{ has no } c_0\text{-subsequence.} \end{cases}
$$

We say X satisfies the ARP if every bad c_0-array in X satisfies the ARP. Note that if X contains a bad c_0-array and satisfies the ARP, then X fails (S). Indeed if X contains a bad c_0-array then by a standard diagonal argument it contains a bad c_0-array which is basic in some order. The sequence (y_n) given in (ARP) is thus seminormalized and weakly null. Proposition 3.2 yields that if X has (S) but fails (US) then X contains a bad c_0-array. Thus Theorem 3.1 will follow from

THEOREM 3.3. *Every Banach space satisfies the ARP.*

The proof requires several steps which we now state as two propositions and a corollary.

PROPOSITION 3.4. *Let (X_n) be a sequence of Banach spaces each of which satisfies the ARP. Let (x_i^n) be a bad c_0-array in some Banach space X and for $m \in \mathbf{N}$ set $X^m = [(x_i^n) : i \in \mathbf{N},\ n \geqq m]$. Suppose that for all $m \in M$ there is a bounded linear operator $T_m : X^m \to X_m$ with $\| T_m \| \leqq 1$, such that $(T_m x_i^m)_{i=1}^{\infty}$ is an m-bad c_0-sequence in X_m. Then (x_i^n) satisfies the ARP.*

COROLLARY 3.5. *If (X_n) is a sequence of Banach spaces satisfying the ARP, then $(\Sigma X_n)_{c_0}$ satisfies the ARP. In particular if K is a countable compact metric space, then $C(K)$ satisfies the ARP.*

PROPOSITION 3.6. *Let (x_i^n) be a bad c_0-array such that $(x_i^n)_{i=1}^{\infty}$ is an M_n-bad c_0-sequence for all n. Then there exists a subarray (y_i^n) of (x_i^n) and w^*-compact countable subsets $K_n \subseteq Ba((Y^n)^*)$ (where $Y^n = [y_i^m : m \geqq n, i \in \mathbf{N}]$) such that for all $n \in \mathbf{N}$, $(y_i^n |_{K_n})_{i=1}^{\infty}$ is an $M_n/6$-bad c_0-sequence in $C(K_n)$.*

Assuming these three results we give the

PROOF OF THEOREM 3.3. Let (x_i^n) be a bad c_0-array in X. By passing to a subarray, if necessary, we may assume that for $n \in \mathbf{N}$, $(x_i^n)_{i=1}^{\infty}$ is an M_n-bad c_0-sequence with $M_n > 6n$. By Proposition 3.6 there exists a subarray (y_i^n) and w^*-compact countable sets $K_n \subseteq Ba((Y^n)^*)$ such that $(y_i^n |_{K_n})_{i=1}^{\infty}$ is an n-bad c_0-sequence. Define $T_n : Y^n \to C(K_n)$ by $T_n y = y |_{K_n}$ for $y \in Y^n$. By

Corollary 3.5, $C(K_n)$ satisfies the ARP and thus by Proposition 3.4, (y_i^n) satisfies the ARP.

It remains to prove 3.4, 3.5 and 3.6.

PROOF OF PROPOSITION 3.4. If there exists $m \in \mathbf{N}$ and a subarray (y_i^n) of (x_i^n) such that $(T_m(y_i^n))_{n,i}$ is a bad c_0-array in X_m, then the fact that the ARP works for $(T_m(y_i^n))_{n,i}$ yields that the ARP works for (y_i^n). Thus by passing to a subsequence of $(x_i^n)_i$, for each n, we may assume (by Proposition 3.2) that

$$(3.1) \qquad \begin{cases} \text{for } m \in \mathbf{N} \text{ there exists } M_m < \infty \text{ such that} \\ \left\| \sum_{i \in F} T_m x_i^n \right\| \leqq M_m \text{ for all } n > m \text{ and finite } F \subseteq \mathbf{N}. \end{cases}$$

We shall inductively choose $(m_n) \in [\mathbf{N}]$ and a subarray (y_i^n) of (x_i^n), with $(y_i^n)_i = (x_i^{m_n})_i$ for all n, reals $a_n > 0$ with $\Sigma_{n=1}^{\infty} a_n \leqq 1$ and a sequence of reals $(N_n)_{n=1}^{\infty}$ such that for all n:

(i) $(T_{m_n}(y_i^n))_{i=1}^{\infty}$ is an m_n-bad c_0-sequence in X_{m_n}.

(ii) $\| \Sigma_{i \in F} y_i^n \| \leqq N_n$ for finite $F \subseteq \mathbf{N}$.

(iii) $a_n m_n > n$.

(iv) $\Sigma_{j=1}^{n-1} a_j N_j < a_n m_n / 4$.

(v) $\Sigma_{j=n+1}^{\infty} a_j M_{m_n} < a_n m_n / 4$.

(vi) $\| \Sigma_{i \in F} T_{m_n}(y_i^l) \| \leqq M_{m_n}$ for $l > n$ and finite $F \subseteq \mathbf{N}$.

First note that (i) and (vi) will be automatically satisfied by the hypothesis of the proposition and (3.1). To start let $a_1 = \frac{1}{2}$ and choose $m_1 \in \mathbf{N}$ such that $a_1 m_1 > 1$. This defines $(y_i^1)_i = (x_i^{m_1})_i$ and since (y_i^1) is a c_0-sequence we can choose N_1 to satisfy (ii) for $n = 1$. The only condition remaining to be satisfied for $n = 1$ is (v) and this will hold provided we require $a_j M_{m_1} < 2^{-j} a_1 m_1 / 4$ for $j > 1$.

Let $n > 1$ and suppose that $(a_j)_{j=1}^{n-1}$, $(m_j)_{j=1}^{n-1}$ and $(N_j)_{j=1}^{n-1}$ have been chosen to satisfy (ii), (iii) and (iv) for "n" replaced by any integer less than n and in addition for $2 \leqq j < n$,

$$(3.2) \qquad 0 < a_j < \min\{2^{-j}, 2^{-j} a_k m_k / 4 M_{m_k} : 1 \leqq k < j\}.$$

Choose $a_n > 0$ to satisfy (3.2) for "j" replaced by "n". Then choose $m_n \in \mathbf{N}$, $m_n > m_{n-1}$, such that (iii) and (iv) hold. Choose N_n so that (ii) holds. This completes the induction. Note that by (3.2), (v) holds for all n and $\Sigma_{j=1}^{\infty} a_j \leqq 1$.

Let (y_k) be given by $y_k = \Sigma_{j=1}^{\infty} a_j y_k^j$ and let (y_{k_i}) be a subsequence of (y_k). We shall show that $\sup_l \| \Sigma_{i=1}^{l} y_{k_i} \| = \infty$ and thus (y_k) has no c_0-subsequence. Fix n and by (i) choose l_n such that $\| \Sigma_{i=1}^{l_n} T_{m_n}(y_{k_i}^n) \| > m_n$. Thus

$$\left\| \sum_{i=1}^{l_n} y_{k_i} \right\| \geqq \left\| \sum_{i=1}^{l_n} \sum_{j=n}^{\infty} a_j y_{k_i}^j \right\| - \left\| \sum_{i=1}^{l_n} \sum_{j=1}^{n-1} a_j y_{k_i}^j \right\|$$

$$\geqq \left\| T_{m_n}\left(\sum_{i=1}^{l_n} \sum_{j=n}^{\infty} a_j y_{k_i}^j \right) \right\| - \sum_{j=1}^{n-1} a_j \left\| \sum_{i=1}^{l_n} y_{k_i}^j \right\|$$

$$\geqq a_n \left\| \sum_{i=1}^{l_n} T_{m_n}(y_{k_i}^n) \right\| - \sum_{j=n+1}^{\infty} a_j \left\| \sum_{i=1}^{l_n} T_{m_n}(y_{k_i}^j) \right\| - \sum_{j=1}^{n-1} a_j N_j \quad \text{(by (ii))}$$

$$\geqq a_n m_n - \sum_{j=n+1}^{\infty} a_j M_{m_n} - a_n m_n/4$$

(by the choice of l_n, (iv) and (vi))

$$\geqq a_n m_n - a_n m_n/4 - a_n m_n/4 \quad \text{(by (v))}$$

$$= a_n m_n/2.$$

Since n was arbitrary and $a_n m_n \to \infty$ by (iii), this completes the proof. ■

PROOF OF COROLLARY 3.5. Let (x_i^n) be a bad c_0-array in $X = (\Sigma X_n)_{c_0}$ and let R_m denote the natural projection of X onto X_m.

CLAIM. For all $M < \infty$ there exists $m, n \in \mathbf{N}$ and a subsequence (y_i) of $(x_i^n)_{i=1}^{\infty}$ such that $(R_m(y_i))_{i=1}^{\infty}$ is an M-bad c_0-sequence.

Indeed if the claim is false we obtain, by Proposition 3.2, that there exists $M < \infty$ such that for all $m, n \in \mathbf{N}$ every subsequence of $(x_i^n)_{i=1}^{\infty}$ contains a further subsequence (y_i) with $\| \Sigma_{i \in F} R_m(y_i) \| \leqq M$ for all finite $F \subseteq \mathbf{N}$. Fix n such that $(x_i^n)_{i=1}^{\infty}$ is an $(M+3)$-bad c_0-sequence. By a gliding hump argument choose a subsequence (y_i) of $(x_i^n)_i$ and $(m_i) \in [\mathbf{N}]$ such that for all $i \in \mathbf{N}$:

(i) $\sup_{m > m_i} \| R_m y_i \| \leqq i^{-1}$,

(ii) $\sup_{m \in \{1, \dots, m_i\}} \| \Sigma_{j=i+1}^{p} R_m y_j \| \leqq M$ for $p > i$.

Let $p \in \mathbf{N}$ and choose $m \in (m_{i-1}, m_i]$ for some $i \in \mathbf{N}$ ($m_0 = 0$) such that

$$\left\| \sum_{j=1}^{p} y_j \right\| = \left\| \sum_{j=1}^{p} R_m(y_j) \right\| .$$

Now

$$\left\| \sum_{j=1}^{p} R_m(y_j) \right\| \leqq \left\| \sum_{j=1}^{i-1} R_m(y_j) \right\| + \| R_m(y_i) \| + \left\| \sum_{j=i+1}^{p} R_m(y_j) \right\|$$

(where we make the obvious adjustments if $p \leqq i$). Thus by (i) and (ii)

$$\left\| \sum_{j=1}^{p} y_j \right\| \leqq (i-1)(i-1)^{-1} + 1 + M = M + 2.$$

This contradicts the fact that $(x_i^n)_i$ is an $(M+3)$-bad c_0-sequence and establishes the claim.

By the claim we can choose an increasing sequence of integers $N(n))_{n=1}^{\infty}$, a sequence of integers $(M(n))_{n=1}^{\infty}$ and subsequences $(y_i^n)_i \subseteq (x_i^{N(n)})_{i=1}^{\infty}$ such that $(R_{M(n)}(y_i^n))_i$ is an n-bad c_0-sequence for all n. Letting

$$T_n = R_{M(n)} \big|_{[y_i^r]_{i \in \mathbf{N}, r \geqq n}}$$

we see that the hypothesis of Proposition 3.4 is satisfied (for (x_i^n) replaced by (y_i^n) and X_n replaced by $X_{M(n)}$) and thus (y_i^n), and hence (x_i^n), satisfies the ARP. This proves the first assertion of the corollary.

If K is a countable compact limit ordinal α and $\beta_n \uparrow \alpha$, then $C(\alpha) \sim (\Sigma\, C(\beta_n))_{c_0}$. Thus by induction we see that $C(\alpha)$ satisfies the ARP for all such α. In view of the isomorphic classification of $C(K)$ for K countable compact metric (see [BP2]) this completes the proof. ■

Proof of Proposition 3.6. The array (x_i^n) satisfies $1 \geqq \| x_i^n \| \geqq \inf_j \| x_j^n \| > 0$ for each $n, i \in \mathbf{N}$. Since for each n, $(x_i^n)_{i=1}^{\infty}$ is weakly null, by passing to subsequences using the standard diagonal argument we may assume (x_i^n) is basic. Moreover we may assume our array is now labeled in triangular fashion $(x_i^n)_{1 \leqq n \leqq i}$ and is basic in the lexicographical order with "first letter" i and "second letter" n. (Thus the order is $x_1^1, x_2^1, x_2^2, x_3^1, x_3^2, \ldots$.) By renorming we may assume (x_i^n) is a monotone basis in this order.

It suffices to find for all n, a subsequence $({}^1y_i^n)$ of (x_i^n) and a w^*-compact countable set ${}^1K_1 \subseteq Ba({}^1Y^*)$ $({}^1Y = [({}^1y_i^n)]_{1 \leqq n \leqq i})$ such that $({}^1y_i^n \mid {}^1K_1)_i$ is an $M_1/6$-bad c_0-sequence in $C({}^1K_1)$. Indeed if this can be done, then we repeat the process inductively to further trim $({}^1y_i^n)_{2 \leqq n \leqq i}$ and obtain $({}^2y_i^n)_{2 \leqq n \leqq i}$ and ${}^2K_2$ etc. The array $(y_i^n)_{n \leqq i}$ which satisfies the conclusion of the proposition is then given by $(y_i^n)_{i=n}^{\infty} = ({}^ny_i^n)_{i=n}^{\infty}$ and $K_n \equiv {}^nK_n \big|_{[(y_i^m)]_{n \leqq m \leqq i}}$. Of course each K_n is a quotient of ${}^nK_n$ an thus is still countable and w^*-compact. Having said all this we shall simplify the notation by writing (y_i^n) and K_1 in place of $({}^1y_i^n)$ and ${}^1K_1$ respectively.

LEMMA 3.7. *There exists* $(l_i) \in [\mathbf{N}]$ *and finite sets* $F_i^n \subseteq [-1, 1]$ *with the following properties. If* $y_i^n = x_{l_i}^n$ *for* $1 \leqq n \leqq i$ *and if* $k_1 < \cdots < k_p$ *are given such that* $\| \Sigma_{i=1}^p y_{k_i}^1 \| > M_1$, *then there exists* $f \in 3Ba(Y^*)$, *where* $Y = [(y_i^n)_{1 \leqq n \leqq i}]$, *such that*

$$\text{(A)} \qquad \begin{cases} \text{(a) } \Sigma_{i=1}^p f(y_{k_i}^1) > M_1/2, \\ \text{(b) } f(y_i^n) \in F_i^n \quad \text{for } n \leqq i, \\ \text{(c) } f(y_i^n) = 0 \quad \text{if } i \notin \{k_1, \dots, k_p\}. \end{cases}$$

Let us assume the lemma and show how to construct K_1 with the desired properties. Let

$$\mathscr{K} = \left\{ (k_1, \dots, k_p) : \left\| \sum_{i=1}^r y_{k_i}^1 \right\| \leqq M_1 \text{ for all } r < p \text{ and } \left\| \sum_{i=1}^p y_{k_i}^1 \right\| > M_1 \right\}.$$

Clearly $\mathscr{K}$ is countable and moreover $\bar{\mathscr{K}}$, the closure of $\mathscr{K}$ in $2^{\mathbf{N}}$, contains only finite sets. Indeed if $(k_i) \in [\mathbf{N}] \cap \bar{\mathscr{K}}$, then for all $p \in \mathbf{N}$, $\{k_i\}_{i=1}^p$ is a proper initial segment of an element of $\mathscr{K}$. In particular $\| \Sigma_1^p y_{k_i}^1 \| \leqq M_1$ which contradicts that (x_i^1) is a M_1-bad c_0-sequence.

For each element $(k_i)_1^p \in \mathscr{K}$, choose an element $f \in 3BaY^*$ which satisfies (A) of Lemma 3.7. Let $q = f/3$ and $G_i^n = \frac{1}{3} F_i^n$ for $n \leqq i$. We let $\mathscr{G}$ be the set of all such g's. Note that for $g \in \mathscr{G}$, $g(y_i^n) \in G_i^n$. For $m \geqq 0$ let Q_m be the basis projection of Y onto $[(y_i^n) ; 1 \leqq n \leqq i \leqq m]$. Of course (y_i^n) is also a monotone basis in the lexicographic order, and so $\| Q_m \| \leqq 1$. Let

$$K_1 = \{Q_m^* g : g \in \mathscr{G}, m \geqq 0\}.$$

Clearly K_1 is a countable subset of BaY^* and by (a) of (A) $(y_i^1 |_{K_1})$ is an $M_1/6$-bad c_0-sequence.

It remains only to check that K_1 is w^*-compact. Let $(k_n) \subseteq K_1$ be w^*-convergent to $k \in Ba(Y^*)$. Let $k_n = Q_{m_n}^* g_n$ for some $m_n \in \mathbf{N}$ and $g_n \in \mathscr{G}$, and suppose that g_n was derived from a set $A_n \in \mathscr{K}$. By passing to a subsequence we may assume that $A_n \to A \in \bar{\mathscr{K}}$. As we noted A must be finite. We may assume (g_n) is w^*-convergent to $g \in Ba(Y^*)$. By (b) and (c), if $q = \max A$, $Q_q^* g_n = Q_q^* g = g$ for large n. Thus $g \in K_1$. We may assume $m_n \to m$ or diverges to ∞. If $m_n \to \infty$ or $m \geqq q$, then $k = g \in K_1$. Otherwise $Q_{m_n}^* g_n \to Q_m^* g = k$ and since $Q_m^* g \in K_1$, $k \in K_1$. ∎

The proof of Lemma 3.7 will make repeated use of the following generalization of a result of Elton ([E], see also [O, Lemma 4.6]).

LEMMA 3.8. *Let $(x_i^n)_{1 \leqq n \leqq i}$ be an array in X such that for all n, $(x_i^n)_{i=n}^\infty$ is weakly null. Let $B \subseteq Ba(X^*)$. Then for all $\varepsilon > 0$, $C < \infty$, $n \in \mathbf{N}$ and $N \in [\mathbf{N}]$ there exists $L \in [N]$ such that if $(l_i)_0^p \subseteq L$ with $n \leqq l_0 < l_1 < l_2 < \cdots < l_p$ and if there exists $f \in B$ with $\Sigma_{j=1}^p f^+(x_{l_j}^1) > C$, then there exists $g \in B$ with $\Sigma_{j=1}^p g^+(x_{l_j}^1) > C$ and $|g(x_{l_0}^m)| < \varepsilon$ for $1 \leqq m \leqq n$.*

PROOF. For $p \in \mathbf{N}$ let $\mathscr{A}_p = \{I \in [\mathbf{N}] : I = (i_j)_{j=0}^\infty,\ i_0 \geqq n$ and if there exists $f \in B$ with $\Sigma_{j=1}^p f^+(x_{i_j}^1) > C$ then there exists $g \in B$ with $\Sigma_{j=1}^p g^+(x_{i_j}^1) > C$ and $|g(x_{i_0}^m)| < \varepsilon$ for $1 \leqq m \leqq n\}$. Let $\mathscr{A} = \bigcap_{p=1}^\infty \mathscr{A}_p$. Each $\mathscr{A}_p$ is closed in $[\mathbf{N}]$ whence so is $\mathscr{A}$. In particular $\mathscr{A}$ is Ramsey and so there exists $L \in [N]$ with $[L] \subseteq \mathscr{A}$ or $[L] \subseteq [\mathbf{N}] \setminus \mathscr{A}$. If $[L] \subseteq \mathscr{A}$ we are done and thus suppose $[L] \subseteq [\mathbf{N}] \setminus \mathscr{A}$. Let $L = (l_j)_{j=0}^\infty$ and fix $p \in \mathbf{N}$. For $q \leqq p$ let $L_q = \{l_q, l_{p+1}, l_{p+2}, \ldots\}$. $L_q \notin \mathscr{A}$ and thus $L_q \notin \mathscr{A}_{r_q}$ for some r_q. Thus there exists $f_q \in B$ with $\Sigma_{j=1}^{r_q} f_q^+(x_{l_{p+j}}^1) > C$ and if $g \in B$ with $\Sigma_{j=1}^{r_q} g^+(x_{l_{p+j}}^1) > C$ then for some $1 \leqq m \leqq n$, $|g(x_{l_q}^m)| \geqq \varepsilon$.

Choose q_0 such that $r_{q_0} = \min\{r_q : 1 \leqq q \leqq p\}$. Thus

$$C < \sum_{j=1}^{r_{q_0}} f_{q_0}^+(x_{l_{p+j}}^1) \leqq \sum_{j=1}^{r_q} f_{q_0}^+(x_{l_{p+j}}^1) \qquad \text{for } 1 \leqq q \leqq p.$$

Hence for $1 \leqq q \leqq p$ there exists $1 \leqq m_q \leqq n$ with $|f_{q_0}(x_{l_q}^{m_q})| \geqq \varepsilon$. Let $g_p \equiv f_{q_0}$ and let $g \in Ba(X^*)$ be a w^*-limit point of $(g_p)_1^\infty$. It follows that for $q \in \mathbf{N}$ there exists $1 \leqq \bar{m}_q \leqq n$ with $|g(x_{l_q}^{\bar{m}_q})| > \varepsilon$ and hence one of the n sequences, $(x_{l_q}^m)_{q=1}^\infty$, $1 \leqq m \leqq n$, is not weakly null, a contradiction. ■

TERMINOLOGY. We shall say L is obtained from $(B, \varepsilon, C, n, N)$ by Lemma 3.8.

PROOF OF LEMMA 3.7. Let $\varepsilon = \min\{1, M_1/4\}$ and let $(b_i^n)_{1 \leqq n \leqq i}$ be the biorthogonal functionals to the monotone basis $(x_i^n)_{1 \leqq n \leqq i}$ of X. For $1 \leqq n \leqq i$ choose $\varepsilon_i^n > 0$ such that

$$\sum_{i=1}^\infty \sum_{n=1}^i \varepsilon_i^n \, \| b_i^n \| < \varepsilon. \tag{3.3}$$

Let H_i^n be a finite ε_i^n-net in $[-1, 1]$ with $0 \in H_i^n$ for each $1 \leqq n \leqq i$. Define $B^1 = \{f \in 2Ba(X^*) : f(x_i^n) \in H_i^n$ for $1 \leqq n \leqq i\}$. Observe that by (3.3) given $g \in Ba(X^*)$ there exists $f \in B^1$ with $|f(x_i^n) - g(x_i^n)| < \varepsilon_i^n$ for all $1 \leqq n \leqq i$. In particular if $g(\Sigma_{j \in F} x_j^1) > M_1$ for some finite $F \subseteq \mathbf{N}$, then $f(\Sigma_{j \in F} x_j^1) > 3M_1/4$.

Choose $\varepsilon_m > 0$ so that

$$(3.4)\qquad \sum_{m=1}^{\infty} m\varepsilon_m \sup\{\|b_j^n\| : 1 \leqq n \leqq j \text{ and } n \leqq m\} < \varepsilon.$$

Note that the "sup" in (3.4) is finite since for all n, $(x_j^n)_{j=n}^{\infty}$ is seminormalized. For $m \in \mathbf{N}$, let $\{C_1^m, \ldots, C_{p(m)}^m\}$ be an $\varepsilon_m/2$-net in $(0, M_1]$. Let L_1^1 be obtained from $(B^1, \varepsilon_1, C_1^1, 1, \mathbf{N})$ by Lemma 3.8. Let $L_2^1 \in [L_1^1]$ be obtained from $(B^1, \varepsilon_1, C_2^1, 1, L_1^1)$ by 3.8. Continue until we obtain $L_1 \equiv L_{p(1)}^1$ from $(B^1, \varepsilon_1, C_{p(1)}^1, 1, L_{p(1)-1}^1)$ by 3.8, and define $l_1 = \min L_1$. This defines $y_1^1 = x_{l_1}^1$ and we let $F_1^1 \equiv H_{l_1}^1$.

For the second step (to obtain l_2) we partition B^1 into finitely many sets

$$B_t^2 = \{f \in B^1 : f(y_1^1) = t\}, \qquad t \in F_1^1.$$

We apply Lemma 3.8 repeatedly to $(B_t^2, \varepsilon_2, C_q^2, 2, L)$ beginning with $L = L_1$ and letting $t \in F_1^1$ and $1 \leqq q \leqq p(2)$ vary independently over all possibilities. At each application L will be the subsequence of L_1 obtained from the previous step. Let L_2 be the last sequence obtained and choose $l_2 \in L_2$ with $l_2 > l_1$. This defines $y_2^n = x_{l_2}^n$ and $F_2^n = H_{l_2}^n$ for $n = 1, 2$.

Let us briefly outline the induction step. Assume $l_1 < l_2 < \cdots < l_m$ and L_m have been chosen in the manner now described. This defines $y_i^n = x_{l_i}^n$ and $F_i^n = H_{l_i}^n$ for $1 \leqq n \leqq i \leqq m$. For every $\vec{t} = (t_i^n) \in \Pi_{1 \leqq n \leqq i \leqq m} F_i^n$ we set

$$B_{\vec{t}}^{m+1} = \{f \in B^1 : f(y_i^n) = t_i^n,\ 1 \leqq n \leqq i \leqq m\}.$$

This partitions B^m into finitely many sets. We then apply Lemma 3.8 repeatedly to $(B_{\vec{t}}^{m+1}, \varepsilon_{m+1}, C_q^{m+1}, m+1, L)$, beginning with $L = L_m$, as $\vec{t}$ and q range over all possibilities. We let L_{m+1} be the subsequence ultimately obtained and choose $l_{m+1} \in L_{m+1}$ with $l_{m+1} > l_m$.

Thus (y_i^n) and (F_i^n) have been chosen such that

$$(\mathrm{B})\qquad \begin{cases} \text{given } n < k_1 < \cdots < k_p, \text{ if there exists } f \in B^1 \text{ with} \\ \displaystyle\sum_{i=1}^{p} f^+(y_{k_i}^1) > C_q^n \text{ for some } 1 \leqq q \leqq p(n), \\ \text{then there exists } g \in B^1 \text{ with} \\ (\mathrm{a}')\ \displaystyle\sum_{i=1}^{p} g^+(y_{k_i}^1) > C_q^n, \\ (\mathrm{b}')\ g(y_i^m) = f(y_i^m) \text{ for } 1 \leqq m \leqq i < n, \\ (\mathrm{c}')\ |g(y_n^m)| < \varepsilon_n \text{ for } 1 \leqq m \leqq n. \end{cases}$$

Let $\| \Sigma_{i=1}^{p} y_{k_i}^1 \| > M_1$. As we noted above, there eixsts $g \in B^1$ with $\Sigma_{i=1}^{p} g^+(y_{k_i}^1) > \frac{3}{4} M_1$. We shall show that (B) implies there exists $h \in B^1$ with

$$\text{(C)} \qquad \begin{cases} \text{(a}'') \ \sum_{i=1}^{p} h^+(y_{k_i}^1) > M_1/2, \\ \text{(b}'') \ |h(y_i^n)| = 0 \quad \text{if } i > k_p \text{ and} \\ \qquad h(y_i^n)| < \varepsilon_i \quad \text{if } i \notin \{k_1, \dots, k_p\} \text{ or} \\ \qquad\qquad \text{if } h(y_i^1) < 0. \end{cases}$$

Assuming (C), let's derive (A). By perturbing h we obtain $f \in X^*$ such that $f(y_i^n) = h(y_i^n)$ if $i \in \{k_1, \dots, k_p\}$ and $h(y_i^1) \geqq 0$ and $f(y_i^n) = 0$ otherwise. From (C) we have

$$\begin{aligned} \| f - h \| &\leqq \sum_{i=1}^{k_p} \varepsilon_i \sum_{n=1}^{i} \| b_{l(i)}^n \| \\ &\leqq \sum_{i=1}^{\infty} \varepsilon_i \cdot i \cdot \sup\{ \| b_j^n \| : 1 \leqq n \leqq j \text{ and } n \leqq i \} \\ &< \varepsilon \leqq 1 \quad \text{by (3.4).} \end{aligned}$$

Thus $\| f \| \leqq \| h \| + 1 \leqq 3$ and clearly f satisfies (A).

It remains to show that (C) holds. Thus let $g \in B^1$ such that $\Sigma_1^p g^+(y_{k_i}^1) > \frac{3}{4} M_1$. We shall apply (B) k_p-times beginning with the function g. To start let $C_0 = \frac{3}{4} M_1$ and $\beta_0 = 0$. Choose $C_1 = C_q^1$ for some $1 \leqq q \leqq p(1)$ such that $0 < C_0 - C_1 < \varepsilon_1$. If $k_1 = 1$ and $g(y_1^1) \geqq 0$ we set $h_1 = g$ and let $\beta_1 = g_1(y_1^1) = h_1^+(y_1^1)$. If $k_1 = 1$ but $g(y_1^1) < 0$ we apply (B) to $1 < k_2 < \cdots < k_p$, g and C_1. This yields $h_1 \in B^1$ with $\Sigma_{i=1}^{p} h_1^+(y_{k_i}^1) > C_1$ and $|h(y_1^1| < \varepsilon_1$. We set $\beta_1 = h_1^+(y_1^1)$. If $k_1 > 1$ we apply (B) to $1 < k_1 < \cdots < k_p$, g and C_1, obtaining $h_1 \in B^1$ with $\Sigma_{i=1}^{p} h_1^+(y_{k_i}^1) > C_1$ and $|h_1(y_1^1)| < \varepsilon_1$. In this case we let $\beta_1 = 0$.

Assume $1 \leqq s < k_p$ and $h_s \in B^1$ and numbers $(\beta_i)_1^s$, $(C_i)_1^s$ have been chosen such that

(i) $0 < (C_{r-1} - \beta_{r-1}) - C_r < \varepsilon_r$ for $1 \leqq r \leqq s$.

(ii) $\Sigma_{\{i : k_i \geqq s\}} h_s^+(y_{k_i}^1) > C_s$.

(iii) If $1 \leqq r \leqq s$ and $r = k_i$ for some $1 \leqq i \leqq p$, then $\beta_r = h_s^+(y_r^1)$, otherwise $\beta_r = 0$.

(iv) $|h_s(y_r^m)| < \varepsilon_r$ for $1 \leqq m \leqq r \leqq s$ provided $r \notin \{k_1, \dots, k_p\}$ or $h_s(y_r^1) < 0$. (Note that by our construction in the first step, all conditions hold for $s = 1$.)

To construct h_{s+1}, we first choose $C_{s+1} = C_q^{s+1}$ for some $1 \leqq q \leqq p(s+1)$ so that $0 < (C_s - \beta_s) - C_{s+1} < \varepsilon_{s+1}$, thus satisfying (i) for $s+1$. (If $0 \leqq C_s - \beta_s < \varepsilon_{s+1}$ we set $h = Q_s^* h_s$ and note that the estimates below show that h satisfies (C). If $s+1 = k_j$ for some $1 \leqq j \leqq p$ and $h_s(y_{s+1}^1) \geqq 0$ we let $h_{s+1} = h_s$ and $\beta_{s+1} = h_{s+1}^+(y_{s+1}^1)$. Thus (iii) and (iv) hold for $s+1$. To see (ii) for $s+1$, we observe that (by (ii) and (i) for s)

$$\sum_{\{i:k_i \geqq s+1\}} h_{s+1}^+(y_{k_i}^1) = \sum_{\{i:k_i \geqq s\}} h_s^+(y_{k_i}^1) - \beta_s > C_s - \beta_s > C_{s+1}.$$

If $s+1 = k_j$ for some $1 \leqq j \leqq p$ and $h_s(y_{s+1}^1) < 0$ we apply (B) to $s+1 < k_{j+1} < k_{j+2} < \cdots < k_p$, h_s and C_{s+1} to obtain $h_{s+1} \in B^1$.

Note that (B) applies in this setting since

$$\sum_{i=j+1}^{p} h_s^+(y_{k_i}^1) = \sum_{\{i:k_i \geqq s+1\}} h_s^+(y_{k_i}^1)$$

$$= \sum_{\{i:k_i \geqq s\}} h_s^+(y_{k_i}^1) - \beta_s > C_s - \beta_s > C_{s+1}.$$

We then let $\beta_{s+1} = h_{s+1}^+(y_{s+1}^1)$. By (a′) of (B) we have

$$\sum_{\{i:k_i \geqq s+1\}} h_{s+1}^+(y_{k_i}^1) = h_{s+1}^+(y_{s+1}^1) + \sum_{i=j+1}^{p} h_{s+1}^+(y_{k_i}^1) > C_{s+1}$$

and thus (ii) holds for $s+1$. (b′) and (c′) of (B) imply that (iv) is fulfilled for $s+1$ and (iii) holds trivially.

Finally if $s+1 \notin \{k_1, \ldots, k_p\}$, say $k_{j-1} < s+1 < k_j$ ($k_0 = 0$), we apply (B) to $s+1 < k_j < \cdots < k_p$, h_s and C_{s+1}. Note that (B) applies since again

$$\sum_{i=j}^{p} h_s^+(y_{k_i}^1) = \sum_{\{i:k_i \geqq s\}} h_s^+(y_{k_i}^1) - \beta_s > C_s - \beta_s > C_{s+1}.$$

We let $\beta_{s+1} = 0$ and thus the new function h_{s+1} satisfies (iii) for $s+1$. (ii) holds for $s+1$ by (a′) of (B) and (iv) holds easily by (b′) and (c′) of (B).

The construction is complete. Let $h = Q_{k_p}^* h_{k_p}$ and we verify that h satisfies (C). $h(y_i^n) = 0$ if $i > k_p$) and the remaining conditions of (b″) hold by (iv) for h_{k_p}. It remains to show that (a″) holds or equivalently that

$$\sum_{i=1}^{p} h_{k_p}^+(y_{k_i}^1) > M_1/2.$$

Now

$$\begin{aligned}
\sum_{i=1}^{p} h_{k_p}^{+}(y_{k_i}^{1}) &= \sum_{i=1}^{p} \beta_{k_i} \quad \text{(by (iii))} \\
&= \sum_{r=1}^{k_p} \beta_{r-1} + \beta_{k_p} \geqq \sum_{r=1}^{k_p} (C_{r-1} - C_r - \varepsilon_r) + \beta_{k_p} \quad \text{(by (i))} \\
&= C_0 - C_{k_p} - \sum_{r=1}^{k_p} \varepsilon_r + \beta_{k_p} \\
&\geqq C_0 - \sum_{r=1}^{\infty} \varepsilon_r \quad \text{(observing that } \beta_{k_p} \geqq C_{k_p} \text{ by (ii))} \\
&\geqq \tfrac{3}{4}M_1 - \tfrac{1}{4}M_1 = M_1/2,
\end{aligned}$$

since by (3.4), $\Sigma_1^\infty \varepsilon_r < \varepsilon \leqq M_1/4$. ■

4. Duality

The natural dual analogue of property (S) (respectively, (US)) is the Schur property (respectively, strong Schur property). A Banach space X has the *Schur property* if given $\delta > 0$ every sequence $(x_n) \subseteq Ba(X)$ with $\| x_n - x_m \| \geqq \delta$ for $n \neq m$ admits a subsequence which is C-equivalent to the unit vector basis of l_1 for some C. If $C = 2K\delta^{-1}$ with K independent of δ and the particular sequence (x_n) we say that X has the *K-strong Schur property* [R2]. With the help of Theorem 3.1 we can strengthen a result of [H] to the following

PROPOSITION 4.1. *Let X be a Banach space not containing l_1. If X has property (S), then X^* has the strong Schur property.*

PROOF. Let $(f_n) \subseteq Ba(X^*)$ with $\| f_n - f_m \| > \delta$ for $n \neq m$. By passing to a subsequence we may assume that (f_n) is w^*- convergent to some $f \in Ba(X^*)$. Let $g_n = f_n - f$. It follows from Theorem 3.1 and the proof of theorem 1(e) in [H] that there is a constant C such that some subsequence of (g_n) is $2C\delta^{-1}$-equivalent to the unit vector basis of l_1. Indeed let K be as in formula (2.1). (g_n) is w^*-null and we may suppose $\| g_n \| > \delta/2$ for all n. Choose $(x_n) \subseteq Ba(X)$ with $g_n(x_n) > \delta/2$ for all n. By passing to subsequences we may assume (x_n) is weak Cauchy and satisfies (2.1). Let $\varepsilon > 0$ be arbitrary. By passing to subsequences and the standard perturbation argument we also may assume $g_{2n}(x_{2n} - x_{2n+1}) > \frac{1}{2}\delta - \varepsilon$ for all n and $g_{2n}(x_{2m} - x_{2m+1}) = 0$ for all $m \neq n$. It follows that for $(a_i) \subseteq \mathbf{R}$,

$$\left\| \sum a_i g_{2i} \right\| \geqq K^{-1}(\tfrac{1}{2}\delta - \varepsilon) \sum |a_i|.$$

From the following proposition we deduce that X^* has the $(K+\eta)$-strong Schur property for all $\eta > 0$. ■

Proposition 4.2. *Let (x_i) be a sequence in a Banach space X satisfying $\| \Sigma a_i x_i \| \geqq \eta \Sigma |a_i|$ for all $(a_i) \subseteq \mathbf{R}$ and some $\eta > 0$. Let $x \in X$. Then for some $N \in \mathbf{N}$,*

$$\left\| \sum_{i=N+1}^{\infty} a_i(x_i + x) \right\| \geqq \eta \sum_{i=N+1}^{\infty} |a_i|$$

for all $(a_i) \subseteq \mathbf{R}$.

Proof. We can assume (or else we can take $N = 0$) that there exists $N \in \mathbf{N}$ and scalars $(b_i)_{i=1}^N$ with $\Sigma_{i=1}^N b_i = 1$ and

$$\left\| \sum_{i=1}^{N} b_i(x_i + x) \right\| < \eta \sum_{i=1}^{N} |b_i|. \tag{4.1}$$

Let $(a_i) \subseteq \mathbf{R}$ and set $A = \Sigma_{i=N+1}^{\infty} a_i$. Thus

$$\begin{aligned}\left\| \sum_{i=N+1}^{\infty} a_i(x_i + x) \right\| &\geqq \left\| (-A) \sum_{i=1}^{N} b_i x_i + \sum_{i=N+1}^{\infty} a_i x_i \right\| - \left\| A \sum_{i=1}^{N} b_i(x_i + x) \right\| \\ &\geqq \eta \left(|A| \sum_{i=1}^{N} |b_i| + \sum_{i=N+1}^{\infty} |a_i| \right) - |A| \eta \sum_{i=1}^{N} |b_i| \\ &\qquad\qquad \text{(using the hypothesis and (4.1))} \\ &= \eta \sum_{i=N+1}^{\infty} |a_i|.\end{aligned}$$

■

Remark 4.3. (1) The analogue of Theorem 3.1 is false, even for dual spaces. Indeed using an example of J. Lindenstrauss (cf. [JO]), let X_n be a sequence space equipped with the norm

$$\| (a_i) \|_n = \sup \left\{ \sum_{i \in F} |a_i| : F \subseteq \mathbf{N} \text{ and } |F| \leqq n \right\}.$$

It is easy to see that if $X = (\Sigma_{n=1}^{\infty} X_n)_{c_0}$, X^* has the Schur property, while failing the strong Schur property.

(2) One might also wish to consider generalizations of Theorem 3.1 to l_p $(1 < p < \infty)$. Let us say that a Banach space X has *property* (S_p) if every weakly null normalized sequence in X has a subsequence K-equivalent to the unit vector basis of l_p for some K. X has *property* (US_p) if K is independent of the particular sequence. These properties have been studied for subspaces X of L_p.

If X is a subspace of L_p $(2 < p < \infty)$ and X has (S_p) then X has (US_p) and moreover X embeds into l_p [JO]. However for $1 < p < 2$ there exists $X \subseteq L_p$ with (S_p) but not (US_p) [JO]. Johnson [J] has shown that if $X \subseteq L_p$ has (US_p) then X embeds into l_p.

Added in proof. The authors have proved the following generalization of Theorem 3.1: *Let X be a Banach space, $1 \leqq p < \infty$, such that every weakly null sequence in $Ba(X)$ admits a subsequence with a C-upper l_p estimate for some C. Then C can be chosen independent of the sequence.*

References

[BP1] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Stud. Math. **17** (1958), 151–164.

[BP2] C. Bessaga and A. Pełczyński, *Spaces of continuous functions IV*, Stud. Math. **19** (1960), 53–62.

[C] P. Cembranos, *The hereditary Dunford–Pettis-property on $C(K, E)$*, Ill. J. Math. **31** (1987), 365–373.

[D] J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, New York, 1984.

[E] J. Elton, Thesis, Yale University, New Haven, CT.

[GP] F. Galvin and K. Prikry, *Borel sets and Ramsey's theorem*, J. Symbolic Logic **38** (1973), 193–198.

[H] J. Hagler, *A counterexample to several questions about Banach spaces*, Stud. Math. **60** (1977), 289–307.

[HOR] R. Haydon, E. Odell and H. Rosenthal, *On certain subclasses of Baire-1 functions and their applications to Banach space theory*, in preparation.

[J] W. B. Johnson, *On quotients of L_p which are quotients of l_p*, Compos. Math. **34** (1977), 69–89.

[JO] W. B. Johnson and E. Odell, *Subspaces of L_p which embed into l_p*, Compos. Math. **28** (1974), 37–49.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, Berlin, 1979.

[O] E. Odell, *Applications of Ramsey theorems to Banach space theory*, Univ. of Texas Press, Austin, 1981.

[P] A. Pełczyński, *A connection between weakly unconditional convergence and weakly completeness of Banach spaces*, Bull. Acad. Polon. Sci. **6** (1958), 251–253.

[R] H. P. Rosenthal, *A characterization of Banach spaces containing l_1*, Proc. Natl. Acad. Sci. U.S.A. **71** (1974), 2411–2413.

[R2] H. P. Rosenthal, Seminar Notes, University of Paris.