ON $c_0$ SEQUENCES IN BANACH SPACES

BY

H. KNAUST\textsuperscript{t} AND E. ODELL

Department of Mathematics, The University of Texas at Austin,
Austin, TX 78712, USA

ABSTRACT
A Banach space has property (S) if every normalized weakly null sequence contains a subsequence equivalent to the unit vector basis of $c_0$. We show that the equivalence constant can be chosen "uniformly", i.e., independent of the choice of the normalized weakly null sequence. Furthermore we show that a Banach space with property (S) has property (u). This solves in the negative the conjecture that a separable Banach space with property (u) not containing $l_1$ has a separable dual.

1. Introduction

A Banach space $X$ is said to have property (S) if every normalized weakly null sequence in $X$ admits a subsequence which is $C$-equivalent to the unit vector basis of $c_0$ for some $C < \infty$. If the constant $C$ is independent of the particular sequence we say $X$ has uniform (S) or (US). A second property relating the internal structure of a Banach space to that of $c_0$ is property (u). One way of formulating this property is to say $X$ has property (u) if whenever $(x_n)$ is a weak Cauchy but not weakly convergent sequence in $X$, there exists $(y_n)$, a block basis of convex combinations of $(x_n)$, which is equivalent to the summing basis for $c_0$.

The definition of property (u) is due to A. Pełczyński [P]. He defined the property as follows. If $x^{**} \in X^{**}$ is the $w^*$-limit of a sequence in $X$ then there exists $(y_n) \subseteq X$, which converges $w^*$ to $x^{**}$ and satisfies

$$\sum_{n=1}^{\infty} |x^*(y_{n+1}) - x^*(y_n)| < \infty \quad \text{for all} \quad x^* \in X^*.$$ 

\textsuperscript{t} This is part of this author's Ph.D. dissertation prepared at The University of Texas at Austin under the supervision of H. P. Rosenthal.

Research partially supported by NSF Grant DMS-8601752.

Received October 7, 1988
(In the terminology of [HOR], \(B_i(X) \subseteq \text{DBSC}(X)\).) The equivalence of our definition and Pełczyński's was noted in [HOR] and follows easily from the fact that if \((x_n) \subseteq X\) also converges \(\omega^*\) to \(x^{**}\), then \(\text{dist}(\text{conv}(x_n), \text{conv}(y_n)) = 0\). By [BP1] and [R], if \(X\) has property \((u)\) and \(Y\) is any infinite dimensional subspace of \(X\), then \(Y\) is reflexive or contains \(c_0\) or \(l_1\). Since every subspace of a space with unconditional basis has property \((u)\) [P], it was conjectured by J. Hagler that if \(X\) is a separable space with property \((u)\) and not containing \(l_1\), then \(X^*\) is separable (see [H]).

In §2 we prove that property \((S)\) implies property \((u)\). In view of the tree space \(JH\) constructed by Hagler [H] this yields a negative answer to the conjecture. Indeed Hagler showed \(JH\) has property \((S)\), does not contain \(l_1\) and has nonseparable dual.

Property \((S)\) was considered by P. Cembranos in [C]. It was noted to be equivalent to the "hereditary Dunford Pettis property": every (infinite dimensional) subspace of \(X\) has the Dunford Pettis property. This equivalence follows easily from the deep "nearly unconditional" theorem (Theorem 2.4 below) of J. Elton ([E]; see also [O]). The question whether \((S)\) implies \((US)\) is raised in [C] (and was originally brought to our attention by A. Pełczyński). We show this to be true in §3. Part of our argument requires a generalization of Elton's argument for the aforementioned theorem.

A corollary of our two main results (see Corollary 2.3) is that \(X\) has property \((S)\) iff there exists \(C < \infty\) so that whenever \((x_n) \subseteq Ba(X)\) is weak Cauchy, there exists a subsequence \((x'_n)\) with

\[
\sum_{n=1}^{\infty} |x^*(x'_{n+1}) - x^*(x'_n)| \leq C \quad \text{for all } x^* \in Ba(X^*).
\]

(Equivalently)

\[
\left\| \sum_{n=1}^{k} e_n(x'_{n+1} - x'_n) \right\| \leq C \quad \text{for all } k \text{ and } e_i = \pm 1.
\]

This contrasts nicely with property \((u)\) which may be described similarly except that \((x'_n)\) is not necessarily a subsequence of \((x_n)\) but rather a block basis of convex combinations of \((x_n)\).

We use standard Banach space terminology as may be found in the books [LT] or [D]. The proofs of both our main results require some Ramsey theory (as can be found in [O], [LT] or [D]). The summing basis for \(c_0\) is the basis \((s_n)\) given by \(s_n = \sum_{i=1}^{n} e_i\), where \((e_i)\) is the unit vector basis
of $c_0$. Finally, it is perhaps worth noting that $l_1$ has property $(S)$ and by [R], if $X$ has property $(S)$, then every infinite dimensional subspace of $X$ contains $l_1$ or $c_0$. Both properties $(S)$ and $(u)$ are hereditary (the later case is due to Pełczyński [P]).

We wish to thank H. Rosenthal for useful discussions regarding this paper.

2. Property $(S)$ implies Property $(u)$

**Theorem 2.1.** If $X$ has property $(S)$, then $X$ has property $(u)$.

We first review the Ramsey theorem we require. If $M$ is an infinite subsequence of $N$, $[M]$ denotes the set of all (infinite) subsequences of $M$. $\tau$ is the pointwise topology on $[N]$, i.e., the relative topology of $[N] \subseteq 2^N$, given the product topology. $\mathcal{A} \subseteq [N]$ is said to be Ramsey if for all $M \in [N]$ there exists $L \in [M]$ such that either $[L] \subseteq \mathcal{A}$ or $[L] \subseteq [N] \setminus \mathcal{A}$. It is known that if $\mathcal{A}$ is $\tau$-Borel then $\mathcal{A}$ is Ramsey [GP]. For a proof of this result, some history and more general results see [O].

**Proof of Theorem 2.1.** Let $(x_n) \subseteq Ba(X)$ be weak Cauchy but not weakly convergent. By passing to a subsequence we may assume that $(x_n)$ is basic and moreover $(y_n)$ is seminormalized basic where $y_n \equiv x_{n+1} - x_n$ [BP1]. (Since $X$ has property $(S)$, we could have also assumed, by passing to a subsequence, that $(y_{2n})$ or $(y_{2n-1})$ is equivalent to the unit vector basis of $c_0$. If we could obtain this simultaneously for both sequences, we would be finished and this is where Ramsey theory enters.)

For $k$ and $K \in \mathbb{N}$ define

$$\mathcal{A}_k(K) = \left\{ M \in [N]: M = (m_i) \text{ satisfies } \left\| \sum_{i=1}^{k} \epsilon_i (x_{m_i} - x_{m_{i+1}}) \right\| \leq K \right\}.$$ 

$\mathcal{A}_k(K)$ is $\tau$-closed and thus $\mathcal{A}(K) \equiv \bigcap_{k=1}^{\infty} \mathcal{A}_k(K)$ is also $\tau$-closed and $\mathcal{A} \equiv \bigcup_{k=1}^{\infty} \mathcal{A}_k(K)$ is $\tau$-Borel. Consequently $\mathcal{A}$ is Ramsey. Choose $M = (m_i) \in [N]$ so that either $[M] \subseteq \mathcal{A}$ or $[M] \subseteq [N] \setminus \mathcal{A}$. Since $X$ has property $(S)$ we obtain $[M] \subseteq \mathcal{A}$. Thus $M \in \mathcal{A}(K_1)$ and $(m_i)^\infty_{i=2} \in \mathcal{A}(K_2)$ for some $K_1, K_2$. It follows that for $x^* \in Ba(X^*)$, 
\[ \sum_{i=1}^{\infty} |x^*(x_{m_{i+1}}) - x^*(x_{m_i})| = \sum_{i=1}^{\infty} |x^*(x_{m_i}) - x^*(x_{m_{i-1}})| + \sum_{i=1}^{\infty} |x^*(x_{m_{i+1}}) - x^*(x_{m_i})| \]
\[ \leq K_1 + K_2. \]

In Pełczyński's terminology [P], \( \Sigma x_{m_i} \) is a w.u. C. In particular \( (x_{m_{i+1}} - x_{m_i})_{i=1}^{\infty} \) is equivalent to the unit vector basis of \( c_0 \) and so \( (x_{m_i}) \) is equivalent to the summing basis for \( c_0 \).

**Remark 2.2.** If \( X \) has property \((US)\), the proof yields a fixed \( K \) satisfying: if \( (x_n) \) is a weak Cauchy sequence in \( Ba(X) \) then there exists a subsequence \( (x_{m_i}) \) with

\[ \sum_{i=1}^{\infty} |x^*(x_{m_{i+1}}) - x^*(x_{m_i})| \leq K \quad \text{for all } x^* \in Ba(X^*). \]

In fact this turns out to be an equivalence.

**Corollary 2.3.** \( X \) has property \((US)\) iff there exists \( K < \infty \) such that if \( (x_n) \subseteq Ba(X) \) is weak Cauchy, then there exists a subsequence \( (x_{m_i}) \) of \( (x_n) \) satisfying (2.1).

The proof requires Elton's nearly unconditional theorem which we first recall.

**Theorem 2.4 (Elton [E]).** For \( 0 < \delta \leq 1 \) there exists a constant \( K(\delta) < \infty \) such that if \( (x_n) \) is a normalized weakly null sequence in a Banach space, then there exists a basic subsequence \( (x^*_i) \) with the following property. If \( (a_i) \subseteq \mathbb{R} \) with \( |a_i| \leq 1 \) for all \( i \), and \( F \subseteq \{ i : |a_i| \geq \delta \} \), then

\[ \left\| \sum_{i \in F} a_i x^*_i \right\| \leq K(\delta) \left\| \sum_{i=1}^{\infty} a_i x^*_i \right\|. \]

**Proof of Corollary 2.3.** By Remark 2.2 it suffices to show that \( X \) has property \((US)\) if it satisfies the condition in the corollary. Let \( (x_n) \) be a normalized weakly null sequence in \( X \) which satisfies both the conclusion of Theorem 2.4 and condition (2.1). We may assume that \( 2^{-1} \sup |a_i| \leq \| \sum a_i x_i \| \) and thus we have \( (x_{2n}) \) is \( 2 \cdot K \cdot K(1) \)-equivalent to the unit vector basis of \( c_0 \). Indeed if \( F \subseteq \mathbb{N} \) is finite, then by (2.2) and (2.1)
\[ \left\| \sum_{n \in F} x_{2n} \right\| \leq K(1) \quad \left\| \sum_{n \in F} (x_{2n} - x_{2n-1}) \right\| \leq K \cdot K(1). \]

3. **Property (S) implies Property (US)**

**Theorem 3.1.** If \( X \) has property (S) then \( X \) has property (US).

One's first thoughts on this theorem are that it is false. The counterexample should be \( X = (\Sigma X_n)_{c_0} \) where the \( X_n \)'s are a sequence of bad \( c_0 \)'s (e.g., \( X_n = C(\omega^n) \)). However it is easy to construct in such a space a normalized weakly null sequence without a \( c_0 \)-subsequence. Theorem 3.1 is proved by showing that this construction can be carried out in general. We give some definitions to make this precise.

A sequence \((x_i)\) in a Banach space \( X \) is called a \( c_0 \)-sequence if \( \| x_i \| \leq 1 \) for all \( i \) and \((x_i)\) is equivalent to the unit vector basis of \( c_0 \). For \( M < \infty \) we say \((x_i)\) is an \( M \)-bad \( c_0 \)-sequence if \((x_i)\) is a \( c_0 \)-sequence with the additional property that for all subsequences \((x'_i)\) of \((x_i)\) there exists \( k \in \mathbb{N} \) such that \( \| \sum_{j=1}^{k} x'_j \| > M \). The following proposition, due to W. B. Johnson (see [O]), yields that if \( X \) has property (S) but fails to have (US), then \( X \) contains \( M \)-bad \( c_0 \)-sequences for all \( M \).

**Proposition 3.2.** Let \((x_i)\) be a \( c_0 \)-sequence and let \( M < \infty \). Then there exists a subsequence \((x'_i)\) of \((x_i)\) such that either

(a) \((x'_i)\) is an \( M \)-bad \( c_0 \)-sequence, or

(b) \( \| \sum_{i \in F} x'_i \| \leq M \) for all finite \( F \subseteq \mathbb{N} \).

**Proof.** Let
\[ \mathcal{A} = \left\{ L = (l_j) \in [\mathbb{N}] : \left\| \sum_{j=1}^{k} x_{l_j} \right\| \leq M \text{ for all } k \in \mathbb{N} \right\}. \]
\( \mathcal{A} \) is \( \tau \)-closed and therefore Ramsey. Choose \( L \in [\mathbb{N}] \) such that either \([L] \subseteq [\mathbb{N}] \setminus \mathcal{A} \) or \([L] \subseteq \mathcal{A} \) and let \((x'_i) = (x_i)_{i \in L} \). In the first case we obtain (a) and in the second (b) holds.

We continue with some more definitions. A collection \((x^n_i)_{i,n \in \mathbb{N}} \subseteq X \) is called an array in \( X \). An array \((y^n_p)\) is a subarray of the array \((x^n_i)\) if there exists \((m_n)_{n \in \mathbb{N}} \subseteq [\mathbb{N}] \) such that for all \( n \in \mathbb{N} \), \((y^n_p)_{p \geq 1} \) is a subsequence of \((x^n_{m_p})_{p \geq 1} \). An array \((x^n_i)\) is a bad \( c_0 \)-array if there exists \( M_n \to \infty \) such that for all \( n \in \mathbb{N} \), \((x^n_i)_{i \geq 1} \) is an \( M_n \)-bad \( c_0 \)-sequence.

A bad \( c_0 \)-array \((x^n_i)\) satisfies the array procedure (ARP) if
(ARP)
\[
\begin{align*}
\text{there exists a subarray } (y_i^n) \text{ of } (x_i^n) \text{ and reals } a_n > 0 \text{ with } \sum_{n=1}^{\infty} a_n &\leq 1 \\
\text{such that if } y_i = \sum_{n=1}^{\infty} a_n y_i^n, \text{ then } (y_i) \text{ has no } c_0\text{-subsequence.}
\end{align*}
\]

We say $X$ satisfies the ARP if every bad $c_0$-array in $X$ satisfies the ARP. Note that if $X$ contains a bad $c_0$-array and satisfies the ARP, then $X$ fails $\text{(S)}$. Indeed if $X$ contains a bad $c_0$-array then by a standard diagonal argument it contains a bad $c_0$-array which is basic in some order. The sequence $(y_n)$ given in (ARP) is thus seminormalized and weakly null. Proposition 3.2 yields that if $X$ has $\text{(S)}$ but fails $\text{(US)}$ then $X$ contains a bad $c_0$-array. Thus Theorem 3.1 will follow from

**Theorem 3.3.** Every Banach space satisfies the ARP.

The proof requires several steps which we now state as two propositions and a corollary.

**Proposition 3.4.** Let $(X_n)$ be a sequence of Banach spaces each of which satisfies the ARP. Let $(x_i^n)$ be a bad $c_0$-array in some Banach space $X$ and for $m \in \mathbb{N}$ set $X^m = \{(x_i^n) : i \in \mathbb{N}, \ n \geq m\}$. Suppose that for all $m \in \mathbb{N}$ there is a bounded linear operator $T_m : X^m \to X_m$ with $\|T_m\| \leq 1$, such that $(T_m x_i^m)^{i=1}_{i=m}$ is an $m$-bad $c_0$-sequence in $X_m$. Then $(x_i^n)$ satisfies the ARP.

**Corollary 3.5.** If $(X_n)$ is a sequence of Banach spaces satisfying the ARP, then $(\Sigma X_n)_c$ satisfies the ARP. In particular if $K$ is a countable compact metric space, then $C(K)$ satisfies the ARP.

**Proposition 3.6.** Let $(x_i^n)$ be a bad $c_0$-array such that $(x_i^n)^{i=1}_{i=m}$ is an $M_n$-bad $c_0$-sequence for all $n$. Then there exists a subarray $(y_i^n)$ of $(x_i^n)$ and $w^*$-compact countable subsets $K_n \subseteq Ba((Y^n)^*)$ (where $Y^n = \{y_i^n : m \geq n, i \in \mathbb{N}\}$) such that for all $n \in \mathbb{N}$, $(y_i^n \mid_{K_n})^{i=1}_{i=n}$ is an $M_n/6$-bad $c_0$-sequence in $C(K_n)$.

Assuming these three results we give the

**Proof of Theorem 3.3.** Let $(x_i^n)$ be a bad $c_0$-array in $X$. By passing to a subarray, if necessary, we may assume that for $n \in \mathbb{N}$, $(x_i^n)^{i=1}_{i=n}$ is an $M_n$-bad $c_0$-sequence with $M_n > 6n$. By Proposition 3.6 there exists a subarray $(y_i^n)$ and $w^*$-compact countable sets $K_n \subseteq Ba((Y^n)^*)$ such that $(y_i^n \mid_{K_n})^{i=1}_{i=n}$ is an $n$-bad $c_0$-sequence. Define $T_n : Y^n \to C(K_n)$ by $T_n y = y \mid_{K_n}$ for $y \in Y^n$. By
Corollary 3.5. \( C(K_n) \) satisfies the ARP and thus by Proposition 3.4, \( (y^n_i) \) satisfies the ARP.

It remains to prove 3.4, 3.5 and 3.6.

**PROOF OF PROPOSITION 3.4.** If there exists \( m \in \mathbb{N} \) and a subarray \( (y^n_i) \) of \( (x^n_i) \) such that \( (T_m(y^n_i))_{n,i} \) is a bad \( c_0 \)-array in \( X_m \), then the fact that the ARP works for \( (T_m(y^n_i))_{n,i} \) yields that the ARP works for \( (y^n_i) \). Thus by passing to a subsequence of \( (x^n_i) \), for each \( n \), we may assume (by Proposition 3.2) that

\[
\sum_{i \in F} T_m x^n_i \leq M_m \text{ for all } n > m \text{ and finite } F \subseteq \mathbb{N}.
\]

We shall inductively choose \( (m_n) \in [\mathbb{N}] \) and a subarray \( (y^n_i) \) of \( (x^n_i) \), with \( (y^n_i) = (x^n_i) \), for all \( n \), reals \( a_n > 0 \) with \( \sum_{n=1}^{\infty} a_n \leq 1 \) and a sequence of reals \( (N_n) \) such that for all \( n \):

(i) \( (T_m(y^n_i))_{n,i} \) is an \( m_n \)-bad \( c_0 \)-sequence in \( X_m \).

(ii) \( \| \sum_{i \in F} y^n_i \| \leq N_n \) for finite \( F \subseteq \mathbb{N} \).

(iii) \( a_n m_n > n \).

(iv) \( \sum_{j=1}^{n-1} a_j N_j < a_n m_n / 4 \).

(v) \( \sum_{j=n+1}^{\infty} a_j M_m < a_n m_n / 4 \).

(vi) \( \| \sum_{i \in F} T_m (y^n_i) \| \leq M_m \) for \( l > n \) and finite \( F \subseteq \mathbb{N} \).

First note that (i) and (vi) will be automatically satisfied by the hypothesis of the proposition and (3.1). To start let \( a_1 = \frac{1}{2} \) and choose \( m_1 \in \mathbb{N} \) such that \( a_1 m_1 > 1 \). This defines \( (y^n_1) = (x^n_m) \), and since \( (y^n_1) \) is a \( c_0 \)-sequence we can choose \( N_1 \) to satisfy (ii) for \( n = 1 \). The only condition remaining to be satisfied for \( n = 1 \) is (v) and this will hold provided we require \( a_j M_m, < 2^{-j} a_1 m_1 / 4 \) for \( j > 1 \).

Let \( n > 1 \) and suppose that \( (a_j)_{j=1}^{n-1}, (m_j)_{j=1}^{n-1} \) and \( (N_j)_{j=1}^{n-1} \) have been chosen to satisfy (ii), (iii) and (iv) for "\( n \)" replaced by any integer less than \( n \) and in addition for \( 2 \leq j < n \),

\[
0 < a_j < \min\{2^{-j}, 2^{-j} a_k m_k / 4 M_m \} : 1 \leq k < j \}.
\]

Choose \( a_n > 0 \) to satisfy (3.2) for "\( j \)" replaced by "\( n \)". Then choose \( m_n \in \mathbb{N}, m_n > m_{n-1} \), such that (iii) and (iv) hold. Choose \( N_n \) so that (ii) holds. This completes the induction. Note that by (3.2), (v) holds for all \( n \) and \( \sum_{j=1}^{\infty} a_j \leq 1 \).
Let \((y_k)\) be given by \(y_k = \sum_{j=1}^{\infty} a_j y_j^k \) and let \((y_k)\) be a subsequence of \((y_k)\). We shall show that \(\sup_1 \| \sum_{i=1}^{\infty} y_k \| = \infty\) and thus \((y_k)\) has no \(c_0^\ast\)-subsequence. Fix \(n\) and by (i) choose \(l_n\) such that \(\sum_{i=1}^{l_n} T_{m_i}(y_{k_i}) \geq m_n\). Thus

\[
\left\| \sum_{i=1}^{l_n} y_k \right\| \geq \left\| \sum_{i=1}^{l_n} \sum_{j=n}^{\infty} a_j y_j^k \right\| - \left\| \sum_{i=1}^{l_n} \sum_{j=1}^{n-1} a_j y_j^k \right\|
\]

\[
= \left\| T_{m_n} \left( \sum_{i=1}^{l_n} \sum_{j=n}^{\infty} a_j y_j^k \right) \right\| - \sum_{j=1}^{n-1} a_j \left\| \sum_{i=1}^{l_n} y_j^k \right\|
\]

\[
\geq a_n \left\| \sum_{i=1}^{l_n} T_{m_n}(y_{k_i}) \right\| - \sum_{j=n+1}^{\infty} a_j \left\| \sum_{i=1}^{l_n} T_{m_n}(y_{k_i}) \right\| - \sum_{j=1}^{n-1} a_j N_j \text{ (by (ii))}
\]

\[
\geq a_n m_n - \sum_{j=n+1}^{\infty} a_j M_{m_n} - a_n m_n / 4
\]

(by the choice of \(l_n\), (iv) and (vi))

\[
\geq a_n m_n - a_n m_n / 4 - a_n m_n / 4 \text{ (by (v))}
\]

\[
= a_n m_n / 2.
\]

Since \(n\) was arbitrary and \(a_n m_n \to \infty\) by (iii), this completes the proof. 

**Proof of Corollary 3.5.** Let \((x^p_n)\) be a bad \(c_0^\ast\)-array in \(X = (\Sigma X_n)_{c_0}\) and let \(R_m\) denote the natural projection of \(X\) onto \(X_m\).

**Claim.** For all \(M < \infty\) there exists \(m, n \in \mathbb{N}\) and a subsequence \((y_i)\) of \((x^p_n)_{n=1}^{\infty}\) such that \((R_m(y_i))_{n=1}^{\infty}\) is an \(M\)-bad \(c_0^\ast\)-sequence.

Indeed if the claim is false we obtain, by Proposition 3.2, that there exists \(M < \infty\) such that for all \(m, n \in \mathbb{N}\) every subsequence of \((x^p_n)_{n=1}^{\infty}\) contains a further subsequence \((y_i)\) with \(\| \Sigma_{i \in F} R_m(y_i) \| \leq M\) for all finite \(F \subseteq \mathbb{N}\). Fix \(n\) such that \((x^p_n)_{n=1}^{\infty}\) is an \((M + 3)\)-bad \(c_0^\ast\)-sequence. By a gliding hump argument choose a subsequence \((y_i)\) of \((x^p_n)\), and \((m_i) \in [\mathbb{N}]\) such that for all \(i \in \mathbb{N}\):

(i) \(\sup_{m \geq m_i} \| R_m y_i \| \leq i^{-1}\),

(ii) \(\sup_{m \in \{1, \ldots, m_i\}} \| \Sigma_{j = i+1}^p R_m y_j \| \leq M\) for \(p > i\).

Let \(p \in \mathbb{N}\) and choose \(m \in (m_{i-1}, \ m_i]\) for some \(i \in \mathbb{N}\) \((m_0 = 0)\) such that

\[
\left\| \sum_{j=1}^{p} y_j \right\| = \left\| \sum_{j=1}^{p} R_m(y_j) \right\|.
\]

Now
\[ \left\| \sum_{j=1}^p R_m(y_j) \right\| \leq \left\| \sum_{j=1}^{i-1} R_m(y_j) \right\| + \left\| R_m(y_i) \right\| + \left\| \sum_{j=i+1}^p R_m(y_j) \right\| \]

(where we make the obvious adjustments if \( p \leq i \)). Thus by (i) and (ii)

\[ \left\| \sum_{j=1}^p y_j \right\| \leq (i-1)(i-1)^{-1} + 1 + M = M + 2. \]

This contradicts the fact that \((x^n_i)\) is an \((M + 3)\)-bad \(c_0\)-sequence and establishes the claim.

By the claim we can choose an increasing sequence of integers \( N(n) \), a sequence of integers \((M(n))_{n=1}^\infty\) and subsequences \((y^n_i)_{i=1}^\infty\) such that \((R_{M(n)}(y^n_i))_{i=1}^\infty\) is an \(n\)-bad \(c_0\)-sequence for all \(n\). Letting

\[ T_n = R_{M(n)} \bigg|_{[y^n_i]_{i=N(n)}} \]

we see that the hypothesis of Proposition 3.4 is satisfied (for \((x^n_i)\) replaced by \((y^n_i)\) and \(X_n\) replaced by \(X_{M(n)}\) and thus \((y^n_i)\), and hence \((x^n_i)\), satisfies the ARP. This proves the first assertion of the corollary.

If \(K\) is a countable compact limit ordinal \(\alpha\) and \(\beta_n \uparrow \alpha\), then \(C(\alpha) = (\sum C(\beta_n))_\alpha\). Thus by induction we see that \(C(\alpha)\) satisfies the ARP for all such \(\alpha\). In view of the isomorphic classification of \(C(K)\) for \(K\) countable compact metric (see [BP2]) this completes the proof.

**Proof of Proposition 3.6.** The array \((x^n_i)\) satisfies \(1 \geq \|x^n_i\| \geq \inf \|x^n_i\| > 0\) for each \(n, i \in \mathbb{N}\). Since for each \(n\), \((x^n_i)_{i=1}^\infty\) is weakly null, by passing to subsequences using the standard diagonal argument we may assume \((x^n_i)\) is basic. Moreover we may assume our array is now labeled in triangular fashion \((x^n_i)_{i \leq n \leq i^2}\) and is basic in the lexicographical order with “first letter” \(i\) and “second letter” \(n\). (Thus the order is \(x^n_1, x^n_2, x^n_3, x^n_4, \ldots\).) By renorming we may assume \((x^n_i)\) is a monotone basis in this order.

It suffices to find for all \(n\), a subsequence \((y^n_i)\) of \((x^n_i)\) and a \(w^*\)-compact countable set \(K_i \subseteq B(a)^\infty(1)\) \((1) = \{(1)^n_i \mid i \leq n \leq i^2\}\) such that \((y^n_i) \uparrow K_i\) is an \(M/6\)-bad \(c_0\)-sequence in \(C(1)\). Indeed if this can be done, then we repeat the process inductively to further trim \((y^n_i)_{i \leq n \leq i^2}\) and obtain \((y^n_i)_{i \leq n \leq i^2}\) and \(2K_2\) etc. The array \((y^n_i)_{i \leq n \leq i^2}\) which satisfies the conclusion of the proposition is then given by \((y^n_i)_{i \leq n \leq i^2}\) and \(K_n \equiv \{y^n_i \mid \{i^n_y\}_{i \leq n \leq i^2}\}. Of course each \(K_n\) is a quotient of \(K_n\) an thus is still countable and \(w^*\)-compact. Having said all this we shall simplify the notation by writing \((y^n_i)\) and \(K_i\) in place of \((y^n_i)\) and \(1K_i\) respectively.
LEMMA 3.7. There exists \((l_i) \in \mathbb{N}\) and finite sets \(F_i \subseteq [-1, 1]\) with the following properties. If \(y^n = x^n_i\) for \(1 \leq n \leq i\) and if \(k_1 < \cdots < k_p\) are given such that \(\| \sum_{i=1}^p y_{k_i}^n \| > M_1\), then there exists \(f \in 3Ba(Y^*)\), where \(Y = \{(y^n_i)_{1 \leq n \leq i}\}\), such that

\[
\begin{align*}
(a) & \quad \sum_{i=1}^p f(y_{k_i}^n) > M_1/2, \\
(b) & \quad f(y^n_i) \in F_i \quad \text{for } n \leq i, \\
(c) & \quad f(y^n_i) = 0 \quad \text{if } i \notin \{k_1, \ldots, k_p\}.
\end{align*}
\]

Let us assume the lemma and show how to construct \(K_1\) with the desired properties. Let

\[
\mathcal{H} = \left\{ (k_1, \ldots, k_p) : \left\| \sum_{i=1}^p y_{k_i}^n \right\| \leq M_1 \text{ for all } r < p \text{ and } \left\| \sum_{i=1}^p y_{k_i}^n \right\| > M_1 \right\}.
\]

Clearly \(\mathcal{H}\) is countable and moreover \(\mathcal{H}\), the closure of \(\mathcal{H}\) in \(2^\mathbb{N}\), contains only finite sets. Indeed if \((k_i) \in [\mathbb{N}] \cap \mathcal{H}\), then for all \(p \in \mathbb{N}\), \(\{k_i\}_{i=1}^p\) is a proper initial segment of an element of \(\mathcal{H}\). In particular \(\| \sum_{i=1}^p y_{k_i}^n \| \leq M_1\) which contradicts that \((x_i^r)\) is a \(M_1\)-bad \(c_0\)-sequence.

For each element \((k_i) \in \mathcal{H}\), choose an element \(f \in 3Ba(Y^*)\) which satisfies (A) of Lemma 3.7. Let \(g = f/3\) and \(G_i^n = \frac{1}{n} F_i^n\) for \(n \leq i\). We let \(\mathcal{G}\) be the set of all such \(g\)'s. Note that for \(g \in \mathcal{G}\), \(g(y^n_i) \in G_i^n\). For \(m \geq 0\) let \(Q_m\) be the basis projection of \(Y\) onto \(\{(y^n_i) ; 1 \leq n \leq i \leq m\}\). Of course \((y^n_i)\) is also a monotone basis in the lexicographic order, and so \(\| Q_m \| \leq 1\). Let

\[
K_1 = \{Q^*_m g : g \in \mathcal{G}, m \geq 0\}.
\]

Clearly \(K_1\) is a countable subset of \(Ba(Y^*)\) and by (a) of (A) \((y_i^1 \mid k_i)\) is an \(M_1/6\)-bad \(c_0\)-sequence.

It remains only to check that \(K_1\) is \(w^*\)-compact. Let \((k_n) \subseteq K_1\) be \(w^*\)-convergent to \(k \in Ba(Y^*)\). Let \(k_n = Q^*_m g_n\) for some \(m_n \in \mathbb{N}\) and \(g_n \in \mathcal{G}\), and suppose that \(g_n\) was derived from a set \(A_n \in \mathcal{H}\). By passing to a subsequence we may assume that \(A_n \rightarrow A \in \mathcal{H}\). As we noted \(A\) must be finite. We may assume \((g_n)\) is \(w^*\)-convergent to \(g \in Ba(Y^*)\). By (b) and (c), if \(q = \max A\), \(Q^*_q g_n = Q^*_q g = g\) for large \(n\). Thus \(g \in K_1\). We may assume \(m_n \rightarrow m\) or diverges to \(\infty\). If \(m_n \rightarrow \infty\) or \(m \geq q\), then \(k = g \in K_1\). Otherwise \(Q^*_m g_n \rightarrow Q^*_m g = k\) and since \(Q^*_m g \in K_1, k \in K_1\).

The proof of Lemma 3.7 will make repeated use of the following generalization of a result of Elton ([E], see also [O, Lemma 4.6]).
**Lemma 3.8.** Let \((x^n_i)_{i \leq n \leq i}\) be an array in \(X\) such that for all \(n\), \((x^n_i)_{i \leq n \leq i}\) is weakly null. Let \(B \subseteq \text{Ba}(X^*)\). Then for all \(\varepsilon > 0\), \(C < \infty\), \(n \in \mathbb{N}\) and \(N \subseteq [N]\) there exists \(L \subseteq [N]\) such that if \((l^n_i)_{i \leq n \leq i} \subseteq L\) with \(n \leq l_0 < l_1 < l_2 < \cdots < l_p\) and if there exists \(f \in B\) with \(\Sigma_{j=n}^p f^+(x^n_i) > C\), then there exists \(g \in B\) with \(\Sigma_{j=n}^p g^+(x^n_i) > C\) and \(|g(x^n_m)| < \varepsilon\) for \(1 \leq m \leq n\).

**Proof.** For \(p \in \mathbb{N}\) let \(\mathcal{A}_p = \{I \subseteq [N] : I = (i_j)_{j=0}^\infty, i_0 \geq n\) and if there exists \(f \in B\) with \(\Sigma_{j=n}^p f^+(x^n_i) > C\) then there exists \(g \in B\) with \(\Sigma_{j=n}^p g^+(x^n_i) > C\) and \(|g(x^n_m)| < \varepsilon\) for \(1 \leq m \leq n\}\). Let \(\mathcal{A} = \bigcap_{p=1}^\infty \mathcal{A}_p\). Each \(\mathcal{A}_p\) is closed in \([N]\) whence so is \(\mathcal{A}\). In particular \(\mathcal{A}\) is Ramsey and so there exists \(L \subseteq [N]\) with \([L] \subseteq \mathcal{A}\) or \([L] \subseteq [N] \setminus \mathcal{A}\). If \([L] \subseteq \mathcal{A}\) we are done and thus suppose \([L] \subseteq [N] \setminus \mathcal{A}\). Let \(L = (l^n_i)_{i=0}^\infty\) and fix \(p \in \mathbb{N}\). For \(q \leq p\) let \(L_q = \{l_q, l_{q+1}, l_{q+2}, \ldots\}\). \(L_q \notin \mathcal{A}\) and thus \(L_q \notin \mathcal{A}_{r^q}\) for some \(r^q\). Thus there exists \(f_q \in B\) with \(\Sigma_{j=n}^r f^+(x^n_{l_{q+j}}) > C\) and if \(g \in B\) with \(\Sigma_{j=n}^r g^+(x^n_{l_{q+j}}) > C\) then for some \(1 \leq m \leq n\), \(|g(x^n_m)| \geq \varepsilon\).

Choose \(q_0\) such that \(r_{q_0} = \min\{r_q : 1 \leq q \leq p\}\). Thus
\[
C < \sum_{j=1}^{r_{q_0}} f^+_q(x^n_{l_{q+j}}) \leq \sum_{j=1}^{r_q} f^+_q(x^n_{l_{q+j}}) \quad \text{for } 1 \leq q \leq p.
\]

Hence for \(1 \leq q \leq p\) there exists \(1 \leq m_q \leq n\) with \(|f_q(x^n_{m_q})| \geq \varepsilon\). Let \(g_p \equiv f_q\) and let \(g \in \text{Ba}(X^*)\) be a \(w^*\)-limit point of \((g_p)_{q=0}^\infty\). It follows that for \(q \in \mathbb{N}\) there exists \(1 \leq m_q \leq n\) with \(|g(x^n_{m_q})| > \varepsilon\) and hence one of the \(n\) sequences, \((x^n_{m_q})_{q=0}^\infty\), \(1 \leq m \leq n\), is not weakly null, a contradiction.  

**Terminology.** We shall say \(L\) is obtained from \((B, \varepsilon, C, n, N)\) by Lemma 3.8.

**Proof of Lemma 3.7.** Let \(\varepsilon = \min\{1, M_1/4\}\) and let \((b^n_i)_{i \leq n \leq i}\) be the biorthogonal functionals to the monotone basis \((x^n_i)_{i \leq n \leq i}\) of \(X\). For \(1 \leq n \leq i\) choose \(\varepsilon^n_i > 0\) such that
\[
\sum_{i=1}^{\infty} \sum_{n=1}^i \varepsilon^n_i \|b^n_i\| < \varepsilon.
\]

Let \(H^n_i\) be a finite \(\varepsilon^n_i\)-net in \([-1, 1]\) with \(0 \in H^n_i\) for each \(1 \leq n \leq i\). Define \(B^i = \{f \in 2\text{Ba}(X^*) : f(x^n_i) \in H^n_i \text{ for } 1 \leq n \leq i\}\). Observe that by (3.3) given \(g \in \text{Ba}(X^*)\) there exists \(f \in B^i\) with \(|f(x^n_i) - g(x^n_i)| < \varepsilon^n_i\) for all \(1 \leq n \leq i\). In particular if \(g(\Sigma_{j \in F} x^n_j) > M_1\) for some finite \(F \subseteq \mathbb{N}\), then \(f(\Sigma_{j \in F} x^n_j) > 3M_1/4\).

Choose \(\varepsilon_m > 0\) so that
\[
\sum_{m=1}^{\infty} m e_m \sup \{ \| b_n^m \| : 1 \leq n \leq j \text{ and } n \leq m \} < \varepsilon. 
\]

Note that the "sup" in (3.4) is finite since for all \( n, (x_n^e)_{e=n}^{\infty} \) is seminormalized.

For \( m \in \mathbb{N} \), let \( \{ C_{1}^m, \ldots, C_{p(m)}^m \} \) be an \( \varepsilon_m/2 \)-net in \((0, M_1)\). Let \( L_1^1 \) be obtained from \((B_1, \varepsilon_1, C_1^1, 1, N)\) by Lemma 3.8. Let \( L_2^1 \in [L_1^1] \) be obtained from \((B_1, \varepsilon_1, C_2^1, 1, L_1^1)\) by 3.8. Continue until we obtain \( L_1 \equiv L_{p(1)}^1 \) from \((B_1, \varepsilon_1, C_{p(1)}^1, 1, L_{p(1)-1}^1)\) by 3.8, and define \( l_1 = \min L_1 \). This defines \( y_1^1 = x_1^1 \) and we let \( F_1^1 = H_{l_1}^1 \).

For the second step (to obtain \( l_2 \)) we partition \( B_1 \) into finitely many sets

\[ B_2^1 = \{ f \in B_1 : f(y_1^1) = t \}, \quad t \in F_1^1. \]

We apply Lemma 3.8 repeatedly to \((B_2^1, \varepsilon_2, C_q^2, 2, L)\) beginning with \( L = L_1 \) and letting \( t \in F_1^1 \) and \( 1 \leq q \leq p(2) \) vary independently over all possibilities. At each application \( L \) will be the subsequence of \( L_1 \) obtained from the previous step. Let \( L_2 \) be the last sequence obtained and choose \( l_2 \in L_2 \) with \( l_2 > l_1 \). This defines \( y_2^n = x_2^n \) and \( F_2^n = H_{l_2}^1 \) for \( n = 1, 2 \).

Let us briefly outline the induction step. Assume \( l_1 < l_2 < \cdots < l_m \) and \( L_m \) have been chosen in the manner now described. This defines \( y_i^n = x_i^n \) and \( F_i^n = H_{l_i}^1 \) for \( 1 \leq n \leq i \leq m \). For every \( \tilde{t} = (t_i^n) \in \prod_{1 \leq n \leq i \leq m} F_i^n \) we set

\[ B_{i+1}^m = \{ f \in B_1 : f(y_i^n) = t_i^n, 1 \leq n \leq i \leq m \}. \]

This partitions \( B_m \) into finitely many sets. We then apply Lemma 3.8 repeatedly to \((B_{i+1}^m, \varepsilon_{m+1}, C_q^{m+1}, m + 1, L)\), beginning with \( L = L_m \), as \( t \) and \( q \) range over all possibilities. We let \( L_{m+1} \) be the subsequence ultimately obtained and choose \( l_{m+1} \in L_{m+1} \) with \( l_{m+1} > l_m \).

Thus \((y_i^n)\) and \((F_i^n)\) have been chosen such that

\[
\begin{align*}
given n < k_1 < \cdots < k_p, \text{if there exists } f \in B_1 \text{ with} \\
\sum_{i=1}^{p} f^+(y_{k_i}^1) > C_q^n \text{ for some } 1 \leq q \leq p(n),
\end{align*}
\]

then there exists \( g \in B_1 \) with

\[
\begin{align*}
(a') & \sum_{i=1}^{p} g^+(y_{k_i}^1) > C_q^n, \\
(b') & g(y_{m}^n) = f(y_{m}^n) \text{ for } 1 \leq m \leq i < n, \\
(c') & |g(y_{m}^n)| < \varepsilon_n \text{ for } 1 \leq m \leq n.
\end{align*}
\]
Let $\| \sum_{k=1}^{p} y_k^+ \| > M_1$. As we noted above, there exists $g \in B^1$ with $\sum_{k=1}^{p} g^+(y_k^+) > \frac{1}{2} M_1$. We shall show that (B) implies there exists $h \in B^1$ with

\[
\begin{cases}
(a'') & \sum_{i=1}^{p} h^+(y_k^+) > M_1/2, \\
(b'') & |h(y_i^+)| = 0 \text{ if } i > k_p \text{ and } \\
& |h(y_i^+)| < \epsilon_i \text{ if } i \notin \{k_1, \ldots, k_p\} \text{ or } \\
& \text{if } h(y_i^+) < 0.
\end{cases}
\]

Assuming (C), let's derive (A). By perturbing $h$ we obtain $f \in X^*$ such that $f(y_i^+) = h(y_i^+)$ if $i \in \{k_1, \ldots, k_p\}$ and $h(y_i^+) \geq 0$ and $f(y_i^+) = 0$ otherwise. From (C) we have

\[
\| f - h \| \leq \sum_{i=1}^{k_p} \epsilon_i \sum_{n=1}^{j} \| b_{n(i)}^n \| \\
\leq \sum_{i=1}^{\infty} \epsilon_i \cdot i \cdot \sup\{ \| b_j^n \| : 1 \leq n \leq j \text{ and } n \leq i \} \\
< \epsilon \leq 1 \quad \text{by (3.4)}.
\]

Thus $\| f \| \leq \| h \| + 1 \leq 3$ and clearly $f$ satisfies (A).

It remains to show that (C) holds. Thus let $g \in B^1$ such that $\sum_{k=1}^{p} g^+(y_k^+) > \frac{1}{2} M_1$. We shall apply (B) $k_p$-times beginning with the function $g$. To start let $C_0 = \frac{1}{2} M_1$ and $\beta_0 = 0$. Choose $C_i = C_q$ for some $1 \leq q \leq p(1)$ such that $0 < C_0 - C_1 < \epsilon_i$. If $k_1 = 1$ and $g(y_1^+) \geq 0$ we set $h_1 = g$ and let $\beta_1 = g_1(y_1^+) = h_1^+(y_1^+)$. If $k_1 = 1$ but $g(y_1^+) < 0$ we apply (B) to $1 < k_2 < \cdots < k_p$, $g$ and $C_i$. This yields $h_1 \in B^1$ with $\sum_{k=1}^{p} h_1^+(y_k^+) > C_1$ and $|h_1(y_1^+)| < \epsilon_1$. We set $\beta_1 = h_1^+(y_1^+)$. If $k_1 > 1$ we apply (B) to $1 < k_1 < \cdots < k_p$, $g$ and $C_1$, obtaining $h_1 \in B^1$ with $\sum_{k=1}^{p} h_1^+(y_k^+) > C_1$ and $|h_1(y_1^+)| < \epsilon_1$. In this case we set $\beta_1 = 0$.

Assume $1 \leq s < k_p$ and $h_s \in B^1$ and numbers $(\beta_i)_i$, $(C_i)_i$ have been chosen such that

(i) $0 < (C_{r-1} - \beta_{r-1}) - C_r < \epsilon$, for $1 \leq r \leq s$.

(ii) $\sum_{i: k_i \geq s} h_{r}^+(y_k^+) > C_s$.

(iii) If $1 \leq r \leq s$ and $r = k_i$ for some $1 \leq i \leq p$, then $\beta_r = h_i^+(y_i^+)$, otherwise $\beta_r = 0$.

(iv) $|h_s(y_m^+)| < \epsilon$, for $1 \leq m \leq r \leq s$ provided $r \notin \{k_1, \ldots, k_p\}$ or $h_s(y_r^+) < 0$.

(Note that by our construction in the first step, all conditions hold for $s = 1$.)
To construct $h_{s+1}$, we first choose $C_{s+1} = C_s^{q+1}$ for some $1 \leq q \leq p(s + 1)$ so that $0 < (C_s - \beta_s) - C_{s+1} < \varepsilon_{s+1}$, thus satisfying (i) for $s + 1$. (If $0 \leq C_s - \beta_s < \varepsilon_{s+1}$ we set $h = Q^*_sh_s$ and note that the estimates below show that $h$ satisfies (C). If $s + 1 = k_j$ for some $1 \leq j \leq p$ and $h_s(y_s^{j+1}) \geq 0$ we let $h_{s+1} = h_s$ and $\beta_{s+1} = h_s^{-1}(y_s^{j+1})$. Thus (iii) and (iv) hold for $s + 1$. To see (ii) for $s + 1$, we observe that (by (ii) and (i) for $s$)

$$\sum_{\{i: k_i \geq s + 1\}} h_{s+1}^+(y_i^1) = \sum_{\{i: k_i \geq s + 1\}} h_s^+(y_i^1) - \beta_s > C_s - \beta_s > C_{s+1}.$$

If $s + 1 = k_j$ for some $1 \leq j \leq p$ and $h_s(y_s^{j+1}) < 0$ we apply (B) to $s + 1 < k_{j+1} < k_{j+2} < \cdots < k_p$, $h_s$ and $C_{s+1}$ to obtain $h_{s+1} \in B^1$.

Note that (B) applies in this setting since

$$\sum_{\{i: k_i \geq s + 1\}} h_{s+1}^+(y_i^1) = \sum_{\{i: k_i \geq s + 1\}} h_s^+(y_i^1)$$

$$= \sum_{\{i: k_i \geq s + 1\}} h_s^+(y_i^1) - \beta_s > C_s - \beta_s > C_{s+1}.$$

We then let $\beta_{s+1} = h_{s+1}^+(y_s^{j+1})$. By (a') of (B) we have

$$\sum_{\{i: k_i \geq s + 1\}} h_{s+1}^+(y_i^1) = h_{s+1}^+(y_s^{j+1}) + \sum_{i=j+1}^p h_{s+1}^+(y_i^1) > C_{s+1}$$

and thus (ii) holds for $s + 1$. (b') and (c') of (B) imply that (iv) is fulfilled for $s + 1$ and (iii) holds trivially.

Finally if $s + 1 \notin \{k_1, \ldots, k_p\}$, say $k_{j-1} < s + 1 < k_j (k_0 = 0)$, we apply (B) to $s + 1 < k_{j-1} < \cdots < k_p$, $h_s$ and $C_{s+1}$. Note that (B) applies since again

$$\sum_{\{i: k_i \geq s + 1\}} h_{s+1}^+(y_i^1) = \sum_{\{i: k_i \geq s + 1\}} h_s^+(y_i^1) - \beta_s > C_s - \beta_s > C_{s+1}.$$

We let $\beta_{s+1} = 0$ and thus the new function $h_{s+1}$ satisfies (iii) for $s + 1$. (ii) holds for $s + 1$ by (a') of (B) and (iv) holds easily by (b') and (c') of (B).

The construction is complete. Let $h = Q^*_s h_s$ and we verify that $h$ satisfies (C). $h(y_s^n) = 0$ if $i > k_s$ and the remaining conditions of (b") hold by (iv) for $h_{k_s}$. It remains to show that (a") holds or equivalently that

$$\sum_{i=1}^p h_{k_s}^+(y_i^1) > M_s/2.$$
\[ \sum_{i=1}^{p} h_{k_{p}}^{+}(y_{k_{p}}^{i}) = \sum_{i=1}^{p} \beta_{k_{p}} \quad \text{(by (iii))} \]

\[ = \sum_{r=1}^{k_{p}} \beta_{r-1} + \beta_{k_{p}} \geq \sum_{r=1}^{k_{p}} (C_{r-1} - C_{r} - \epsilon_{r}) + \beta_{k_{p}} \quad \text{(by (i))} \]

\[ = C_{0} - C_{k_{p}} - \sum_{r=1}^{k_{p}} \epsilon_{r} + \beta_{k_{p}} \]

\[ \geq C_{0} - \sum_{r=1}^{\infty} \epsilon_{r} \quad \text{(observing that } \beta_{k_{p}} \geq C_{k_{p}} \text{ by (ii))} \]

\[ \geq \frac{1}{4} M_{1} - \frac{1}{4} M_{1} = M_{1}/2, \]

since by (3.4), \( \sum_{1}^{\infty} \epsilon_{r} < \epsilon \leq M_{1}/4. \)

\section*{4. Duality}

The natural dual analogue of property (S) (respectively, (US)) is the Schur property (respectively, strong Schur property). A Banach space \( X \) has the Schur property if given \( \delta > 0 \) every sequence \( (x_{n}) \subseteq \text{Ba}(X) \) with \( \| x_{n} - x_{m} \| \geq \delta \) for \( n \neq m \) admits a subsequence which is \( C \)-equivalent to the unit vector basis of \( l_{1} \) for some \( C \). If \( C = 2K\delta^{-1} \) with \( K \) independent of \( \delta \) and the particular sequence \( (x_{n}) \) we say that \( X \) has the \( K \)-strong Schur property [R2]. With the help of Theorem 3.1 we can strengthen a result of [H] to the following

\textbf{Proposition 4.1.} Let \( X \) be a Banach space not containing \( l_{1} \). If \( X \) has property (S), then \( X^{*} \) has the strong Schur property.

\textbf{Proof.} Let \( (f_{n}) \subseteq \text{Ba}(X^{*}) \) with \( \| f_{n} - f_{m} \| > \delta \) for \( n \neq m \). By passing to a subsequence we may assume that \( (f_{n}) \) is \( w^{*} \)-convergent to some \( f \in \text{Ba}(X^{*}). \)

Let \( g_{n} = f_{n} - f. \) It follows from Theorem 3.1 and the proof of theorem 1(e) in [H] that there is a constant \( C \) such that some subsequence of \( (g_{n}) \) is \( 2C\delta^{-1} \)-equivalent to the unit vector basis of \( l_{1} \). Indeed let \( K \) be as in formula (2.1). \( (g_{n}) \) is \( w^{*} \)-null and we may suppose \( \| g_{n} \| > \delta/2 \) for all \( n \). Choose \( (x_{n}) \subseteq \text{Ba}(X) \) with \( g_{n}(x_{n}) > \delta/2 \) for all \( n \). By passing to subsequences we may assume \( (x_{n}) \) is weak Cauchy and satisfies (2.1). Let \( \epsilon > 0 \) be arbitrary. By passing to subsequences and the standard perturbation argument we also may assume \( g_{2n}(x_{2n} - x_{2n+1}) > \frac{1}{2}\delta - \epsilon \) for all \( n \) and \( g_{2n}(x_{2m} - x_{2m+1}) = 0 \) for all \( m \neq n \). It follows that for \( (a_{i}) \subseteq \mathbb{R}, \)

\[ \left\| \sum a_{i}g_{2i} \right\| \geq K^{-1}(\frac{1}{2}\delta - \epsilon) \sum |a_{i}|. \]
From the following proposition we deduce that $X^*$ has the $(K + \eta)$-strong Schur property for all $\eta > 0$.

**Proposition 4.2.** Let $(x_i)$ be a sequence in a Banach space $X$ satisfying
\[
\left\| \sum_{i=1}^{\infty} a_i x_i \right\| \leq \eta \left( \sum_{i=1}^{\infty} |a_i|\right)
\]
for all $(a_i) \subseteq \mathbb{R}$ and some $\eta > 0$. Let $x \in X$. Then for some $N \in \mathbb{N},$
\[
\left\| \sum_{i=N+1}^{\infty} a_i (x_i + x) \right\| \leq \eta \left( \sum_{i=N+1}^{\infty} |a_i|\right)
\]
for all $(a_i) \subseteq \mathbb{R}$.

**Proof.** We can assume (or else we can take $N = 0$) that there exists $N \in \mathbb{N}$ and scalars $(b_i)_{i=1}^{N}$ with $\sum_{i=1}^{N} b_i = 1$ and
\[
\left(4.1\right) \quad \left\| \sum_{i=1}^{N} b_i (x_i + x) \right\| < \eta \left( \sum_{i=1}^{N} |b_i|\right).
\]
Let $(a_i) \subseteq \mathbb{R}$ and set $A = \sum_{i=N+1}^{\infty} a_i$. Thus
\[
\left\| \sum_{i=N+1}^{\infty} a_i (x_i + x) \right\| \leq \left\| (A - A) \sum_{i=1}^{N} b_i x_i + \sum_{i=N+1}^{\infty} a_i x_i \right\|
\]
\[
\leq \eta \left( |A| \sum_{i=1}^{N} |b_i| + \sum_{i=N+1}^{\infty} |a_i| \right) - |A| \eta \sum_{i=1}^{N} |b_i|
\]
(using the hypothesis and (4.1))
\[
= \eta \sum_{i=N+1}^{\infty} |a_i|.
\]

**Remark 4.3.** (1) The analogue of Theorem 3.1 is false, even for dual spaces. Indeed using an example of J. Lindenstrauss (cf. [JO]), let $X_n$ be a sequence space equipped with the norm
\[
\| (a_i) \|_n = \sup \left\{ \sum_{i \in F} |a_i| : F \subseteq \mathbb{N} \text{ and } |F| \leq n \right\}.
\]
It is easy to see that if $X = (\sum_{n=1}^{\infty} X_n)_{\ell_1}, X^*$ has the Schur property, while failing the strong Schur property.

(2) One might also wish to consider generalizations of Theorem 3.1 to $l_p$ ($1 < p < \infty$). Let us say that a Banach space $X$ has property $(S_p)$ if every weakly null normalized sequence in $X$ has a subsequence $K$-equivalent to the unit vector basis of $l_p$ for some $K$. $X$ has property $(US_p)$ if $K$ is independent of the particular sequence. These properties have been studied for subspaces $X$ of $L_p$. 
If $X$ is a subspace of $L_p\ (2 < p < \infty)$ and $X$ has ($S_p$) then $X$ has ($US_p$) and moreover $X$ embeds into $l_p\ [JO]$. However for $1 < p < 2$ there exists $X \subset L_p$ with ($S_p$) but not ($US_p$) [JO]. Johnson [J] has shown that if $X \subset L_p$ has ($US_p$) then $X$ embeds into $l_p$.

Added in proof. The authors have proved the following generalization of Theorem 3.1: Let $X$ be a Banach space, $1 \leq p < \infty$, such that every weakly null sequence in $Ba(X)$ admits a subsequence with a $C$-upper $l_p$ estimate for some $C$. Then $C$ can be chosen independent of the sequence.

REFERENCES