Weakly null sequences with upper ℓ_p-estimates

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1. Introduction

A Banach space X has property (S) if every weakly null sequence in $Ba\,X$, the unit ball of X, has a subsequence which is C-dominated by the unit vector basis of c_0 for some constant $C < \infty$. In [**11**] it was shown that if X has property (S), then the constant C can be chosen to be independent of the particular weakly null sequence in $Ba\,X$.

Here we generalize this result to the case of upper ℓ_p-estimates.

Definition 1.1.

Let $1 < p < \infty$. A Banach space X has *property* (S_p) if every weakly null sequence (x_n) has a subsequence (y_n) such that for some constant $C < \infty$,

$$(1) \qquad \left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\| \leq C \text{ for all } (\alpha_n) \in \mathbb{R} \text{ with } \left(\sum_{n=1}^{\infty} |\alpha_n|^p \right)^{1/p} \leq 1 \ .$$

X has *property* (US_p), if there is a constant C such that every normalized weakly null sequence in $Ba\,X$ admits a subsequence (y_n) so that (1) holds. We say that (y_n) has a *C-upper ℓ_p-estimate*, if (1) holds. ∎

Our main result is

Theorem 1. *A Banach space has property (S_p) if and only if it has property (US_p).*

Let us give some examples of Banach spaces which enjoy property (S_p): ℓ_p has property (S_p) $(1 < p < \infty)$. $L_p[0,1]$ has property (S_r), where $r = \min\{2, p\}$. (More generally, every Banach space which has type p and can be embedded into a Banach space with an unconditional basis has property (S_p).) The James space J and its tree version JT have property (S_2) [**1**]. It follows from the results of James [**8**] (see also [**7**]) that every superreflexive Banach space has property (S_p) for some $1 < p < \infty$. Let us note that every subspace of a Banach space X with property (S_p) has property (S_p). If X is in addition reflexive, quotients of X have property (S_p) as well.

A technique employed in the proof of Theorem 1 allows us to strengthen this last result in the following way:

Corollary 1. *Let X be a Banach space with property (S_p) and let Y be a subspace of X not containing ℓ_1. Then the quotient space X/Y has property (US_p).*

Our proof of Theorem 1 is strongly motivated by the arguments in [**11**]. In fact, the proof we present here is valid for the case of property (S) as well (with the usual changes of notation). The key proposition in our proof (Proposition 3.4) is an improvement to our construction in [**11**]. Furthermore we no longer have Johnson's lemma [**11**, Proposition 3.4] at our disposal.

The following remarks were made in [**11**] but seem worth recalling.

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One might ask whether a result like Theorem 1 remains true, if one considers the property that every normalized weakly null sequence in X admits a subsequence which is equivalent to the unit vector basis of ℓ_p.

On the one hand Johnson and the second named author [**9**] have shown this to be false: for $1 < p < 2$ they construct a subspace of $L_p[0,1]$ where each normalized weakly null sequence has a subsequence equivalent to the unit vector basis of ℓ_p, but where the equivalence constant cannot be chosen uniformly for all sequences in question. Their construction can be carried out to produce counterexamples for $p \geq 2$ as well (not within $L_p[0,1]$).

We are indebted to H.P. Rosenthal for pointing out to us that under stronger conditions on the other hand, one obtains the following corollary of Theorem 1:

Corollary 2. *Let X be a Banach space such that X has property (S_p) and X^* has property (S_q) for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the following properties hold for some constant $C < \infty$:*

(i) If ℓ_1 does not embed in X^*, then every normalized weakly null sequence in X contains a subsequence which is C-equivalent to the unit vector basis of ℓ_p and whose closed linear span is C-complemented in X.

(ii) If ℓ_1 does not embed in X, then every normalized weakly null sequence in X^* contains a subsequence which is C-equivalent to the unit vector basis of ℓ_q and whose closed linear span is C-complemented in X^*.

The proof of Theorem 1 will be presented in Section 3. In order to motivate the quite technical proof we present a version of Theorem 1 for spreading models in Section 2. We hope that the spreading-model version (which is quite easy and does not follow from Theorem 1) will give the reader some insight into our approach to the proof of Theorem 1. Section 4 contains the proof of the corollaries; we also state a "weak Cauchy sequence" criterion for property (S_p), due to C. Schumacher [**16**].

Our notation is standard as can be found in [**5**] or [**12**]. If F is a subset in a Banach space X, then $[F]$ denotes the closed linear span of F in X. If L is an infinite subsequence of $\mathbb{N}$, we denote by $\mathcal{P}_\infty(L)$ the set of all infinite subsequences of L. We would like to thank Haskell Rosenthal for his useful suggestions.

2. The spreading model case

Let us recall that a semi-normalized basic sequence (x_n) in a Banach space X is said to have a *spreading model* (e_n), if (e_n) is basic in some Banach space such that for all $k \in \mathbb{N}$ and for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $N < n_1 < n_2 < \ldots < n_k$ and for all scalars (a_i) with $\sup_i |a_i| \leq 1$

$$\left| \; \left\| \sum_{i=1}^{k} a_i e_i \right\| - \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| \; \right| < \varepsilon \; .$$

If (x_n) has a spreading model, the model is unique up to 1-equivalence. Every normalized weakly null sequence admits a subsequence with an unconditional spreading model [**3**], [**4**].

Definition 2.1.

Let $1 < p < \infty$. A Banach space has *property* (M_p) if every normalized weakly null sequence in X admits a subsequence with a spreading model (e_n) which is C-dominated by the unit vector basis of ℓ_p, *i.e.*,

$$\left\| \sum_i a_i e_i \right\| \leq C \text{ for all } (a_i) \in Ba\, \ell_p \; .$$

We say X has *property* (UM_p), if the constant C can be chosen uniformly for all normalized weakly null sequences in X.

We will show the following analogue of Theorem 1:

Proposition 2.2. *A Banach space with property (M_p) has property (UM_p).*

For its proof we will need some technical definitions. Let $1 < p < \infty$ and let X be a Banach space. A normalized weakly null sequence in X is called *good* (respectively C*-good*), if it has a spreading model which satisfies an upper ℓ_p-estimate (respectively a C-upper ℓ_p-estimate). A good sequence in X is called M*-bad*, if its spreading model fails to have an M-upper ℓ_p-estimate. An array $(x_i^n)_{i,n=1}^\infty$ of elements in X is called a *bad array*, if each *column* $(x_i^n)_i$ is an M_n-bad sequence for all $n \in \mathbb{N}$ and $M_n \to \infty$ as $n \to \infty$. An array (y_i^n) is called a *subarray* of an array (x_i^n), if each column of (y_i^n) is a subsequence of $(x_i^{k_n})_i$ for some sequence $k_1 < k_2 < \dots$. Finally let us say a bad array (x_i^n) satisfies the *ℓ_p-array procedure (for spreading models)* if it admits a subarray (y_i^n) and if there are positive scalars a_n with $\sum a_n \le 1$ so that $y_i = \sum_{n=1}^\infty a_n y_i^n$ has no subsequence with a spreading model which has an upper ℓ_p-estimate. (Note that (y_i) is automatically weakly null.)

Proof of Proposition 2.2.

Suppose a Banach space X has property (M_p) but fails property (UM_p). Then X contains a bad array. We will show that every bad array satisfies the ℓ_p-array procedure for spreading models, thus obtaining a contradiction. Indeed, a bad array (x_i^n) contains a (bad) subarray, which is basic in some order (see Lemma 3.7 below). The sequence (y_n) obtained from this subarray by the ℓ_p-array procedure is then semi-normalized and weakly null. The sequence $(y_n/\|y_n\|)$ then has no spreading model with an upper ℓ_p-estimate.

Let (x_i^n) be a bad array in X. We claim that we can find a subarray (y_i^n) of (x_i^n), positive scalars a_n with $\sum a_n \le 1$, $k_N \in \mathbb{N}$ and constants C_N and M_N such that the following properties hold for all $N \in \mathbb{N}$:

$$(y_i^N)_i \text{ is a } C_N\text{-good sequence with its spreading model denoted by } (e_i^N) \tag{2}$$

$$\left\|\sum_{i=1}^{k_N} b_i^N e_i^N\right\| > 2M_N \text{ for some } (b_i^N) \in Ba\,\ell_p \tag{3}$$

$$a_N M_N > N \tag{4}$$

$$\sum_{n=1}^{N-1} a_n C_n < \tfrac{1}{8} a_N M_N \tag{5}$$

$$k_N \sum_{n=N+1}^{\infty} a_n < \tfrac{1}{4} a_N M_N\ . \tag{6}$$

Once the claim has been established we proceed as follows. We set $y_i = \sum_{n=1}^\infty a_n y_i^n$ and let $(y_i)_{i\in L}, L \in \mathcal{P}_\infty(\mathbb{N})$, be a subsequence of (y_i) with spreading model (e_i). Let $C < \infty$ be given. We have to show that (e_i) does not have a C-upper ℓ_p-estimate. By (4) we can choose N so that $a_N M_N > 4C$. Next we choose $j_1 < j_2 < \ldots < j_{k_N}$ in L with j_1 large enough, so that we obtain for all $n \le N$ and all $(c_i) \subseteq \mathbb{R}$ with $|c_i| \le 1$:

$$2\left\|\sum_{i=1}^{k_N} c_i e_i^n\right\| \ge \left\|\sum_{i=1}^{k_N} c_i y_{j_i}^n\right\| \ge \tfrac{1}{2}\left\|\sum_{i=1}^{k_N} c_i e_i^n\right\| \quad\text{and}\quad \left\|\sum_{i=1}^{k_N} c_i y_{j_i}\right\| \le 2\left\|\sum_{i=1}^{k_N} c_i e_i\right\|\ . \tag{7}$$

By (3) we can find $(b_i^N)_{i=1}^{k_N} \in Ba\,\ell_p$ with $\left\|\sum_{i=1}^{k_N} b_i^N e_i^N\right\| > 2M_N$. Thus

$$\begin{aligned}
\left\|\sum_{i=1}^{k_N} b_i^N y_{j_i}\right\| &= \left\|\sum_{i=1}^{k_N} b_i^N \left(\sum_{n=1}^{\infty} a_n y_{j_i}^n\right)\right\| \\
&\geq a_N \left\|\sum_{i=1}^{k_N} b_i^N y_{j_i}^N\right\| - \sum_{n=1}^{N-1} a_n \left\|\sum_{i=1}^{k_N} b_i^N y_{j_i}^n\right\| - \sum_{n=N+1}^{\infty} a_n \left\|\sum_{i=1}^{k_N} b_i^N y_{j_i}^n\right\| \\
&\geq \tfrac{1}{2} a_N \left\|\sum_{i=1}^{k_N} b_i^N e_i^N\right\| - \sum_{n=1}^{N-1} 2a_n \left\|\sum_{i=1}^{k_N} b_i^N e_i^n\right\| - \sum_{n=N+1}^{\infty} a_n \left\|\sum_{i=1}^{k_N} b_i^N y_{j_i}^n\right\| \quad \text{by (7)} \\
&\geq a_N M_N - 2\sum_{n=1}^{N-1} a_n C_N - k_N \sum_{n=N+1}^{\infty} a_n \quad \text{using (2)} \\
&\geq a_N M_N - \frac{a_N M_N}{4} - \frac{a_N M_N}{4} \geq \frac{a_N M_N}{2} \geq 2C \quad \text{by (5) and (6)}\ .
\end{aligned}$$

Consequently $\|\sum_{i=1}^{k_N} b_i^N e_i\| > C$ by (7).

It is left to present the construction of the subarray (y_i^n). Let $a_1 = 1/2$. Since (x_i^n) is a bad array we can find a column n such that the sequence $(x_i^n)_i$ has a subsequence (z_i) which has a spreading model (e_i^1) which fails to satisfy a $2M_1$-upper ℓ_p-estimate, where $M_1 > 2$, *i.e.*, $\|\sum_{i=1}^{k_1} b_i^1 e_i^1\| > 2M_1$ for some $k_1 \in \mathbb{N}$ and some $(b_i^1)_{i=1}^{k_1} \in Ba\,\ell_p$. We set $y_i^1 = z_i$. This defines the first column of the subarray (y_i^n) of (x_i^n). By choosing C_1 large enough we can assure that (y_i^1) is a C_1-good sequence.

We pick $a_2 < 1/4$ such that $a_2 < a_1 M_1/8k_1$, pick (y_i^2), a subsequence of some $(x_i^n)_i$, such that this column is $(2M_2)$-bad, where $a_1 C_1 < a_2 M_2/8$ and $a_2 M_2 > 2$. We choose C_2 so that (y_i^2) is a C_2-good sequence.

If the first $N-1$ columns of the subarray have been chosen in the way just described, we choose

$$0 < a_N < \min_{n=1,\dots,N-1} \left\{ 2^{-N}, \frac{a_n M_n}{4 \cdot 2^{-N} k_n} \right\}\ ,$$

then we choose (y_i^N), again a subsequence of some column of (x_i^n), which is $2M_N$-bad where M_N satisfies (5). Finally k_N is chosen so that $\|\sum_{i=1}^{k_N} b_i^N e_i^N\| > 2M_N$ for some $(b_i^N) \in Ba\,\ell_p$. This completes the induction. It is straightforward to check that (2)–(6) are indeed satisfied. ∎

3. The proof of the main result

The proof for the spreading model version is quite easy, since we can control the length of the sum we consider in the Nth column. This enables us to gain control over the behavior of the columns following the Nth one. Similarly it is straightforward to prove Theorem 1, when we assume additionally that X is embedded into a space with an unconditional basis. The projection onto the Nth column allows us then to preserve the badness of the Nth column without disturbances from the other columns. In the general case we do not know an easy way to similarly gain easy "access" to the Nth column. One step in the proof will be considerations very similar to the ones employed in the proof of the spreading model case (see Lemma 3.6).

We start with some technical definitions analogous to the ones employed in the proof in the last section.

Definitions 3.1.

Let X be a Banach space.

(i) A sequence (x_n) in X is called a *$u\ell_p$-sequence*, if $\|x_n\| \le 1$ for all $n \in \mathbb{N}$, (x_n) converges weakly to 0 and

$$\sup_{(\sum_{n=1}^{\infty}|\alpha_n|^p)^{1/p}\le 1} \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| < \infty \ . \tag{8}$$

(x_n) is called C-$u\ell_p$-sequence, if one can replace (8) by

$$\sup_{(\sum_{n=1}^{\infty}|\alpha_n|^p)^{1/p}\le 1} \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| \le C \ . \tag{9}$$

(ii) A sequence (x_n) in X is called an *M-bad $u\ell_p$-sequence* for a constant $M < \infty$, if (x_n) is a $u\ell_p$-sequence, and no subsequence of (x_n) is an M-$u\ell_p$-sequence. Thus for all subsequences (y_n) there exist $k \in \mathbb{N}$ and $(\alpha_n)_{n=1}^k \in Ba\, \ell_p$ with

$$\left\| \sum_{n=1}^{k} \alpha_n y_n \right\| > M \ .$$

(iii) An array $(x_i^n)_{i,n=1}^{\infty}$ in X is called a *bad $u\ell_p$-array*, if each sequence $(x_i^n)_{i=1}^{\infty}$ is an M_n-bad $u\ell_p$-sequence for some constants M_n with $M_n \to \infty$ as $n \to \infty$.

(iv) (y_i^k) is called a *subarray* of (x_i^n), if there is a subsequence (n_k) of $\mathbb{N}$ such that every sequence $(y_i^k)_{i=1}^{\infty}$ is a subsequence of $(x_i^{n_k})_{i=1}^{\infty}$.

(v) A bad $u\ell_p$-array $(x_i^n)_{i,n=1}^{\infty}$ is said to satisfy the *ℓ_p-array procedure*, if there exists a subarray (y_i^n) of (x_i^n) and there exist $(a_n) \subseteq \mathbb{R}^+$ with $\sum_{n=1}^{\infty} a_n \le 1$ such that the (weakly null) sequence (y_i) with $y_i := \sum_{n=1}^{\infty} a_n y_i^n$ has no $u\ell_p$-subsequence.

An immediate consequence of the definitions are the following observations: A subarray of a bad $u\ell_p$-array is a bad $u\ell_p$-array. A bad $u\ell_p$-array satisfies the ℓ_p-array procedure, if and only if it has a subarray satisfying the ℓ_p-array procedure. The "global" idea for the proof of Theorem 1 can be summarized as follows:

Lemma 3.2. *Let X have property (S_p). Then X has property (US_p) if and only if X satisfies the ℓ_p-array procedure.*

Proof. Clearly if X has property (US_p), X does not contain a bad $u\ell_p$-array and so the ℓ_p-array procedure is satisfied. Let us now assume that X has property (S_p) and satisfies the ℓ_p-array procedure, but fails property (US_p). Thus X contains a bad $u\ell_p$-array. Since X satisfies the ℓ_p-array procedure, we can find a sequence (y_i) in X (by the method described in the array procedure), which does not admit of a $u\ell_p$-subsequence. Observing that (y_i) converges weakly to 0 by the conditions imposed on the a_n's, we see that X fails property (S_p), a contradiction. ∎

The preceding lemma reduces the proof of Theorem 1 to the proof of

Theorem 2. *Every Banach space satisfies the ℓ_p-array procedure.*

The proof of Theorem 2 will be broken up into two major steps. First we show that Theorem 2 is valid for the special case of $C(K)$-spaces where K is a countable compact metric space:

Proposition 3.3. *Let K be a countable compact metric space. Then $C(K)$ satisfies the ℓ_p-array procedure.*

The second major step is to show that the case of a general Banach space can be reduced to the special case of Proposition 3.3:

Proposition 3.4. *Let $(x_i^n)_{i,n=1}^\infty$ be a bad $u\ell_p$-array in a Banach space X. Then there exist a subarray (y_i^n) of (x_i^n) and a countable ω^*-compact subset K of $Ba\,Y^*$, where we set $Y = [y_i^n]_{i,n=1}^\infty$, such that $(y_i^n \mid K)$ is a bad $u\ell_p$-array in $C(K)$.*

Let us observe that Theorem 2 is indeed an easy consequence of the Propositions 3.3 and 3.4. If (x_i^n) is a bad $u\ell_p$-array in X, Proposition 3.4 produces a subarray (y_i^n) and a countable ω^*-compact metric space K in $Ba\,Y^*$ such that $(y_i^n \mid K)$ is a bad $u\ell_p$-array in $C(K)$. Thus there are $(a_n) \subseteq \mathbb{R}^+$ with $\sum_{n=1}^\infty a_n \le 1$ so that the sequence (y_i') in $C(K)$ defined by $y_i' = \sum_{n=1}^\infty a_n y^n_{i\,|\,K}$ has no $u\ell_p$-subsequence. Since $K \subseteq Ba\,Y^*$ it follows that $y_i = \sum_{n=1}^\infty a_n y_i^n$ itself can have no $u\ell_p$-subsequence in X. Thus (x_i^n) satisfies the ℓ_p-array procedure.

We now present the proof of Proposition 3.3. This follows rather easily by induction from the following result.

Lemma 3.5. *Let (X_n) be a sequence of Banach spaces each satisfying the ℓ_p-array procedure. Then $(\sum_{n=1}^\infty X_n)_{c_0}$ satisfies the ℓ_p-array procedure.*

Before we present its proof we need another lemma:

Lemma 3.6. *Let (X_n) be a sequence of Banach spaces each satisfying the ℓ_p-array procedure and let (x_i^n) be a bad $u\ell_p$-array in some Banach space. Set $X = [x_i^n]_{i,n=1}^\infty$ and suppose that for all $m \in \mathbb{N}$ there is a bounded linear operator $T_m : X \to X_m$ with $\|T_m\| \le 1$ such that $(T_m x_i^m)_{i=1}^\infty$ is an m-bad $u\ell_p$-sequence in X_m. Then (x_i^n) satisfies the ℓ_p-array procedure.*

Proof of Lemma 3.6. Let us first consider the (easy) case that there exist an $m \in \mathbb{N}$ and a subarray (y_i^n) of (x_i^n) such that $(T_m y_i^n)_{i,n}$ is a bad $u\ell_p$-array in X_m. Since X_m satisfies the ℓ_p-array procedure, so does $(T_m y_i^n)_{i,n}$. It follows that (y_i^n) satisfies the ℓ_p-array procedure. Thus (x_i^n) itself satisfies the ℓ_p-array procedure.

Now let us assume that the easy case does not apply. By passing to a subarray, if necessary, we can then assume that for all $m \in \mathbb{N}$ there is an integer M_m such that $(T_m x_i^n)_i$ is an M_m-$u\ell_p$-sequence for all $n \in \mathbb{N}$. Inductively we will choose $(m_n) \in \mathcal{P}_\infty(\mathbb{N})$, a subarray $(y_i^n) = (x_i^{m_n})$, $a_n > 0$ with $\sum_{n=1}^\infty a_n \le 1$ and integers (N_n) such that the following properties hold for all $n \in \mathbb{N}$:

$$(T_{m_n}(y_i^n))_i \text{ is an } m_n\text{-bad } u\ell_p\text{-sequence in } X_{m_n} \tag{10}$$

$$(y_i^n)_i \text{ is an } N_n\text{-}u\ell_p\text{-sequence} \tag{11}$$

$$a_n m_n > n \tag{12}$$

$$\sum_{j=1}^{n-1} a_j N_j < a_n m_n/4 \tag{13}$$

$$\sum_{j=n+1}^{\infty} a_j M_{m_n} < a_n m_n/4 \tag{14}$$

$$(15) \qquad \left(T_{m_n} y_i^\ell\right)_i \text{ is an } M_{m_n}\text{-}u\ell_p\text{-sequence for all } \ell \ .$$

Let us note that (10) is automatically fulfilled because of the hypotheses; (15) holds because of our assumption above.

We start the induction as follows. Let $a_1 = 1/2$; choose m_1 such that $a_1 m_1 > 1$ (thus (12) is satisfied for n=1). Since $(y_i^1)_i := (x_i^{m_1})_i$ is a $u\ell_p$-sequence, we can choose an N_1 so that (11) holds. If we require for future a_j's that

$$a_j M_{m_1} < 2^{-j} a_1 m_1/4 \ \text{ for all } \ j > 1 \ ,$$

condition (14) will hold for n=1. The condition (13) does not apply for $n = 1$.

Now let $n > 1$ and suppose that $(a_j)_{j=1}^{n-1}$, $(m_j)_{j=1}^{n-1}$ and $(N_j)_{j=1}^{n-1}$ have been chosen such that (11)–(13) hold for $n-1$ and additionally for all $2 \le j < n$

$$0 < a_j < \min\left\{2^{-j},\, 2^{-j} \frac{a_k m_k}{4\ M_{m_k}} \ : \ 1 \le k < j\right\}$$

Choose a_n such that

$$0 < a_n < \min\left\{2^{-n},\, 2^{-n} \frac{a_k m_k}{4\ M_{m_k}} \ : \ 1 \le k < n\right\} .$$

Then choose m_n large enough to satisfy (12) and (13). This defines $(y_i^n)_i = (x_i^{m_n})_i$. Next choose N_n so that (11) is fulfilled. The induction is complete. Because of the conditions imposed on the a_n's, (14) holds for all $n \in \mathbb{N}$ and $\sum a_n \le 1$.

We set $y_k = \sum_{j=1}^\infty a_j y_k^j$. Let (y_{k_i}) be a subsequence of (y_k). We have to show that (y_{k_i}) is not a $u\ell_p$-sequence, *i.e.*,

$$\sup_{\ell \in \mathbb{N}} \ \sup_{(\alpha_i) \in Ba\, \ell_p^{(\ell)}} \left\| \sum_{i=1}^{\ell} \alpha_i y_{k_i} \right\| = \infty \ .$$

To this end fix n and choose — using (10) — $k \in \mathbb{N}$ and $(\beta_i) \in Ba\, \ell_p$ such that

$$\left\| \sum_{i=1}^{k} T_{m_n}\left(\beta_i y_{k_i}^n\right) \right\| > m_n \ .$$

We obtain the following estimate:

$$\begin{aligned}
\left\| \sum_{i=1}^{k} \beta_i y_{k_i} \right\| &= \left\| \sum_{i=1}^{k} \sum_{j=1}^{\infty} \beta_i a_j y_{k_i}^j \right\| \\
&\ge \left\| \sum_{i=1}^{k} \sum_{j=n}^{\infty} T_{m_n}\left(\beta_i a_j y_{k_i}^j\right) \right\| - \left\| \sum_{i=1}^{k} \sum_{j=1}^{n-1} \beta_i a_j y_{k_i}^j \right\| \\
&\ge \left(a_n \left\| \sum_{i=1}^{k} T_{m_n}\left(\beta_i y_{k_i}^n\right) \right\| - \sum_{j=n+1}^{\infty} a_j \left\| \sum_{i=1}^{k} T_{m_n}\left(\beta_i y_{k_i}^j\right) \right\| \right) - \sum_{j=1}^{n-1} a_j \left\| \sum_{i=1}^{k} \beta_i y_{k_i}^j \right\| \\
&\ge a_n m_n - \sum_{j=n+1}^{\infty} a_j M_{m_n} - \sum_{j=1}^{n-1} a_j N_j \ \text{ by using (10), (15) and (11) resp.} \\
&\ge a_n m_n - a_n m_n/4 - a_n m_n/4 \ \text{ by (13) and (14)} \\
&= a_n m_n/2 \ .
\end{aligned}$$

By (12) $a_n m_n \to \infty$ as $n \to \infty$ and the proof is complete. ∎

Proof of Lemma 3.5. Let (x_i^n) be a bad $u\ell_p$-array in $X = \left(\sum X_n\right)_{c_0}$. We denote by R_m the natural projections $R_m : X \to X_m$. Lemma 3.5 is an easy consequence of the following claim:

> For all $M < \infty$ there are $n, m \in \mathbb{N}$ and a subsequence (y_i) of (x_i^n) such that $(R_m y_i)_i$ is an M-bad $u\ell_p$-sequence.

Assume the claim is false. Thus we can find $M < \infty$ such that for all $m, n \in \mathbb{N}$ every subsequence of $(x_i^n)_{i=1}^\infty$ contains a further subsequence (y_i) such that $(R_m y_i)_i$ is an M-$u\ell_p$-sequence. Fix n such that (x_i^n) is an $(M+3)$-bad $u\ell_p$-sequence . We can find a subsequence (y_i) of (x_i^n) and $(m_i) \in \mathcal{P}_\infty(\mathbb{N})$ such that for all $i \in \mathbb{N}$

$$\sup_{m > m_i} \|R_m y_i\| \le i^{-1} \tag{16}$$

$$(R_m y_j)_{j=i+1}^\infty \text{ is an } M\text{-}u\ell_p\text{-sequence for all } m \le m_i \ . \tag{17}$$

Indeed, set $y_1 = x_1^n$ and choose $m_1 (= 1)$ such that $\|R_m y_1\| \le 1$ for all $m > m_1$. Now pass to a subsequence $(y_{1;\,i})_{i=1}^\infty$ of $(x_i^n)_{i=2}^\infty$ such that $(R_m y_{1;\,j})_{j=1}^\infty$ is an M-$u\ell_p$-sequence for all $m \le m_1$. Set $y_2 = y_{1;\,1}$. Choose m_2 such that $\|R_m y_2\| \le 2^{-1}$ for $m > m_2$. Again by our assumption, we can find a subsequence $(y_{2;\,i})_{i=1}^\infty$ of $(y_{1;\,i})_{i=2}^\infty$ such that $(R_m y_{2;\,j})_{j=1}^\infty$ is an M-$u\ell_p$-sequence for all $m \le m_2$. Set $y_3 = y_{2;\,1}$ and continue in the obvious fashion. The sequence (y_i) we have constructed clearly satisfies (16) and (17).

Since (x_i^n) is an $(M+3)$-bad $u\ell_p$-sequence , we can find $(\alpha_j) \in Ba\,\ell_p$ with

$$\left\| \sum_{j=1}^\infty \alpha_j y_j \right\| > M + 3 \ . \tag{18}$$

On the other hand, by (16) and (17) we obtain the following estimate for $i \in \mathbb{N}$ and $m \in (m_{i-1}, m_i]$ (where $m_0 := 0$) :

$$\begin{aligned} \left\| \sum_{j=1}^\infty R_m(\alpha_j y_j) \right\| &\le \left\| \sum_{j=1}^{i-1} R_m(\alpha_j y_j) \right\| + \|R_m(\alpha_i y_i)\| + \left\| \sum_{j=i+1}^\infty R_m(\alpha_j y_j) \right\| \\ &\le (i-1) \cdot (i-1)^{-1} + 1 + M = M + 2 \ . \end{aligned}$$

Thus we obtain that

$$\left\| \sum_{j=1}^\infty \alpha_j y_j \right\| = \sup_m \left\| R_m \left(\sum_{j=1}^\infty \alpha_j y_j \right) \right\| \le M + 2 \ ,$$

which contradicts (18).

The claim allows us to choose integers $N(1) < N(2) < \dots$ and $(M(n))_{n=1}^\infty$, and subsequences $(y_i^n)_{i=1}^\infty$ of $(x_i^{N(n)})_i$, such that $\left(R_{M(n)} y_i^n\right)_i$ is an n-bad $u\ell_p$-sequence for all $n \in \mathbb{N}$. We let

$$T_n = R_{M(n)|\,[y_i^r]_{i,r=1}^\infty}$$

and apply Lemma 3.6. This finishes the proof. ∎

Proof of Proposition 3.3. Recall that for every countable limit ordinal α we can find a sequence of ordinals $\beta_n < \alpha$, $\beta_n \nearrow \alpha$ such that $C(\alpha)$ is isomorphic to $\left(\sum C(\beta_n)\right)_{c_0}$. Using induction and Lemma 3.5 we obtain that all $C(\alpha)$-spaces, where α is a countable limit ordinal, satisfy the ℓ_p-array procedure. Thus, in view of the isomorphic classification of $C(K)$-spaces for countable compact metric spaces K (see [**2**]), all $C(K)$-spaces for countable compact metric spaces K satisfy the ℓ_p-array procedure. ∎

This completes the first major step. In order to complete the proof of Theorem 2 (and thus the proof of Theorem 1) we have to prove Proposition 3.4.

What is the idea behind its proof? Let $(x_i^n)_{i,n=1}^\infty$ be the given bad $u\ell_p$-array. Let us focus on the nth column, $(x_i^n)_i$. Since $(x_i^n)_{i=1}^\infty$ is a M_n-bad $u\ell_p$-sequence for some constant M_n, we can find for each subsequence (y_i) of $(x_i^n)_{i=1}^\infty$ integers $k_1 < k_2 < \ldots$ such that $\|\sum_{i=1}^\infty \alpha_i y_{k_i}\| > M_n$ for some choice of (α_i) with $\left(\sum_{i=1}^\infty |\alpha_i|^p\right)^{1/p} \le 1$. This badness can be "witnessed" by a functional $f \in Ba\, X^*$: there is an $f \in Ba\, X^*$ such that $f(\sum_{i=1}^\infty \alpha_i y_{k_i}) > M_n$. The countable ω^*-compact metric space K will be formed from a subcollection of the functionals f, which witness the badness of a subsequence of the n-th column of a subarray (y_i^n) of (x_i^n) for some $n \in \mathbb{N}$. While it is not hard to make sure that only countably many functionals are required to sufficiently witness the badness of the array (x_i^n), major difficulties arise from the restriction that K has to be ω^*-compact.

The construction of the "right" functionals will be given in the proof of Proposition 3.8. Before we state this result and present the construction of the subarray (y_i^n) and K needed for Proposition 3.4 we will change the *shape* of the array (x_i^n) to facilitate the quite technical proofs which follow.

Instead of the *square* array $(x_i^n)_{i,n=1}^\infty$ we will use its *triangulated version* $(z_i^n)_{i,n=1}^\infty$, where $z_i^n = x_i^n$ if $n \le i$ and $z_i^n = 0$ otherwise. It is easy to see that a square array satisfies the ℓ_p-array procedure if and only if its triangulated version does. From now on we will drop the zero-entries and just label (x_i^n) in a triangular fashion $(x_i^n)_{1\le n\le i<\infty}$.

We use the triangular labeling to improve the structure of the given bad $u\ell_p$-array (x_i^n) :

Lemma 3.7. *A triangular bad $u\ell_p$-array $(x_i^n)_{n\le i}$ admits a triangular subarray $(y_i^n)_{n\le i}$, which is a basic sequence in its lexicographical order (where i is the first "letter" and n is the second "letter"): $y_1^1, y_2^1, y_2^2, y_3^1, y_3^2, y_3^3, y_4^1, \ldots$.*

Note that we can assume that each column of (x_i^n) is semi-normalized. The proof is then an easy adaptation of the the proof that a normalized weakly null sequence has a basic subsequence, using each column of (x_i^n) is weakly null. We leave the details of the proof of this lemma to the reader.

In the sequel we will assume that the given bad $u\ell_p$-array (x_i^n) is labeled in a triangular fashion and that it is a *bimonotone* basic sequence in its lexicographical order. (Note that we can renorm the underlying Banach space X, since both properties "being a bad $u\ell_p$-array" and "satisfying the ℓ_p-array procedure" are invariant under isomorphisms.) We also assume that $(x_i^n)_i$ is a semi-normalized M_n-bad $u\ell_p$-sequence where $M_n \to \infty$. We will also have occasion to use arrays which are labeled in a trapezoidal manner. For this purpose we define the index set $T_{n_0} = \{(i,n) \mid n \le i \text{ and } i \ge n_0\}$

Proposition 3.8. *Given $n_0 \in \mathbb{N}$ and a trapezoidal subarray of (x_i^n) indexed by T_{n_0} (and again denoted by (x_i^n)), there exist $(\ell_i)_{i=n_0}^\infty \in \mathcal{P}_\infty(\mathbb{N})$ and finite sets $F(i,n) \subseteq [-1,1]$ for $(i,n) \in T_{n_0}$ such that the following holds: If $y_i^n = x_{\ell_i}^n$ for $(i,n) \in T_{n_0}$ and if $n_0 \le k_1 < k_2 < \ldots < k_q$ are given such that $\|\sum_{i=1}^q \beta_i y_{k_i}^{n_0}\| > M_{n_0}$ for some $(\beta_i) \in Ba\,\ell_p$, then there exists an $f \in 3\ Ba\,Y^*$ (where $Y = [y_i^n]_{(i,n)\in T_{n_0}}$) such that*

$$\sum_{i=1}^q f(\alpha_i y_{k_i}^{n_0}) > M_{n_0}/4 \quad \text{for some} \quad (\alpha_i) \in Ba\,\ell_p \tag{19}$$

$$f(y_i^n) \in F(i,n) \quad \text{for all} \quad (i,n) \in T_{n_0} \tag{20}$$

$$f(y_i^n) = 0 \quad \text{for all} \quad (i,n) \in T_{n_0} \quad \text{with} \quad i \notin \{k_1, k_2, \ldots, k_q\}\ . \tag{21}$$

We are now able to complete the proof of Theorem 1 "modulo" the proof of Proposition 3.8.

Proof of Proposition 3.4. We will define the subarray $(y_i^n)_{n\le i}$ by a diagonal procedure using Proposition 3.8 repeatedly. First we apply Proposition 3.8 for $n_0 = 1$ and the array $(x_i^n)_{n\le i}$ and obtain a certain subarray $(z_i^n)_{(i,n)\in T_1}$; we set $y_1^1 = z_1^1$. Next we apply the proposition to $(z_i^n)_{(i,n)\in T_2}$. In return we get a new subarray, which we denote by $(z_i^n)_{(i,n)\in T_2}$. The second row of (y_i^n) is defined by setting $y_2^1 = z_2^1$ and $y_2^2 = z_2^2$.

If the ℓth row of (y_i^n) has been defined (via the array $(z_i^n)_{(i,n)\in T_\ell}$), we apply Proposition 3.8 to the array $(z_i^n)_{(i,n)\in T_{\ell+1}}$. The first row of the array we obtain will be the $(\ell+1)$st row of (y_i^n). This completes the induction. Clearly $(y_i^n)_{n\le i}$ is a subarray of $(x_i^n)_{n\le i}$. Moreover (y_i^n) in its lexicographical order is a subsequence of (x_i^n) in its lexicographical order and thus also bimonotone. Furthermore $(y_i^n)_i$ is a subsequence of $(x_i^n)_i$ for all $n \in \mathbb{N}$.

The set K is constructed next. Let $Y = [y_i^n]_{n \le i}$ and $m \in \mathbb{N}$, and define

$$\mathcal{K}_m = \Bigg\{ (k_1, k_2, \ldots, k_q) \mid \qquad m \le k_1 < k_2 < \ldots < k_q,$$

$$\left\| \sum_{i=1}^{r} \alpha_i y_{k_i}^m \right\| \le M_m \textit{ for all } (\alpha_i) \subseteq Ba\, \ell_p \textit{ for all } r < q$$

$$\textit{and } \left\| \sum_{i=1}^{q} \alpha_i y_{k_i}^m \right\| > M_m \textit{ for some } (\alpha_i) \subseteq Ba\, \ell_p. \Bigg\}$$

Whenever $\vec{k} = (k_1, \ldots, k_q) \in \mathcal{K}_m$, our application of Proposition 3.8 in the definition of the subarray (y_i^n) yields a functional f of norm not greater than 3, which is defined on a certain subspace of X, spanned by some elements of (x_i^n). By first restricting to $[(y_i^n)]_{(i,n)\in T_{n_0}}$ and then extending to Y in a trivial manner, we may assume that $f \in 3\ Ba\, Y^*$ with $\sum_{i=1}^q f(\alpha_i y_{ki}^n) > M_n/4$ for some $(\alpha_i) \in Ba\, \ell_p$ (among its other properties (20) and (21)). It is here, where we use that (y_i^n) is a bimonotone basic sequence. We denote the functional $f/3$ by $f_{\vec{k}}$ and let

$$K_n = \{Q_m^* f_{\vec{k}} \mid m \in \mathbb{N}, \vec{k} \in \mathcal{K}_n\} \ .$$

Here Q_m denotes the natural projection of norm 1 from Y onto $[y_i^n]_{1 \le n \le i \le m}$. Finally we define

$$K = \bigcup_{n=1}^{\infty} K_n \cup \{0\} \ .$$

Let us first note that $(y_{i\,|\,K}^n)_{n \le i}$ is a bad $u\ell_p$-array. Indeed, fix a column n_0. By our construction $(y_i^{n_0})_{i=1}^{\infty}$ is an M_{n_0}-bad $u\ell_p$-sequence . Consequently, given a subsequence $(y_{k_i}^{n_0})$ of $(y_i^{n_0})$, $\vec{k} := (k_1, \ldots, k_q) \in \mathcal{K}_{n_0}$ for some $q \in \mathbb{N}$. By (21) $f_{\vec{k}} = Q_m^* f_{\vec{k}}$ for m large enough and thus $f_{\vec{k}} \in K_{n_0} \subseteq K$. By (19) we obtain that $(y_{i\,|\,K}^{n_0})_{i=1}^{\infty}$ is an $M_{n_0}/12$-bad $u\ell_p$-sequence in $C(K)$.

By the construction of K it is obvious that K is a countable subset of $Ba\, Y^*$. Since Y is separable, K is ω^*-metrizable. It remains to show that K is ω^*-closed.

Let $(g_j) \subseteq K$ and assume that (g_j) converges ω^* to some $g \in Ba\, Y^*$. We have to show that $g \in K$. Every g_j is of the form $Q_{m_j}^* f_{\vec{k}_j}$ for some $m_j \in \mathbb{N}$ and some $\vec{k}_j \in \mathcal{K}_{n_j}$ for some $n_j \in \mathbb{N}$. By passing to a subsequence of the sequence (g_j) we can assume that either $n_j \to \infty$ as $j \to \infty$, or there is an $n \in \mathbb{N}$ such that $n_j = n$ for all $j \in \mathbb{N}$.

Let us first deal with the first alternative. Let i_j be the first element of $\vec{k}_j$. Since $i_j \ge n_j$, $i_j \to \infty$ as $j \to \infty$. Moreover $f_{\vec{k}_j}(y_i^n) = 0$ for all $n \le i < i_j$ by (21). Thus we obtain that $f_{\vec{k}_j} \to 0$ in the ω^*-topology as $j \to \infty$, i. e. $g = 0 \in K$.

From now on we assume that there is an $n \in \mathbb{N}$ such that $\vec{k}_j \in \mathcal{K}_n$ for all $j \in \mathbb{N}$. $\mathcal{K}_n$ is relatively sequentially compact, if endowed with the relative product topology on $\{0,1\}^{\mathbb{N}}$; thus we can assume, by passing to a subsequence of (g_j) if necessary, that $\vec{k}_j \to \vec{k}$ for some $\vec{k} \in \overline{\mathcal{K}_n}$, the closure of $\mathcal{K}_n$.

We claim that $\vec{k}$ is finite. Suppose to the contrary that $\vec{k} = (k_i)_{i=1}^{\infty}$. Since $\vec{k} \in \overline{\mathcal{K}_n}$ we can find for every $q \in \mathbb{N}$ an element in $\mathcal{K}_n$ of the form $(k_1, \ldots, k_q, \ell_1, \ldots, \ell_r)$. By the definition of $\mathcal{K}_n$, $\left\|\sum_{i=1}^q \alpha_i y_{k_i}^n\right\| \le M_n$ for all $(\alpha_i) \subseteq Ba\, \ell_p$. Thus $(y_{k_i}^n)_{i=1}^{\infty}$ is an M_n-$u\ell_p$-sequence, a contradiction to the fact that $(y_i^n)_{i=1}^{\infty}$ is an M_n-bad $u\ell_p$-sequence .

Since $Ba\, Y^*$ is ω^*-sequentially compact, we may additionally assume that $f_{\vec{k}_j}$ converges in the ω^*-sense to some $f \in Ba\, Y^*$. We claim that $f \in K$.

First we observe that $Q_m^* f \in K$ for all $m \in \mathbb{N}$. Why? By (20) and (21) the set $\{Q_m^* f_{\vec{k}_j}(y_i^n) \mid j \in \mathbb{N}, 1 \le n \le i\}$ has only finitely many elements. Since $Q_m^* f_{\vec{k}_j} \stackrel{\omega^*}{\to} Q_m^* f$ as $j \to \infty$, we obtain that $Q_m^* f_{\vec{k}_j} = Q_m^* f$ for

$j \in \mathbb{N}$ large enough; in particular $Q^*_m f \in K$. Next let $q = \max \vec{k}$. Since $\vec{k}_j \to \vec{k}$ and $\vec{k}$ is finite, $Q^*_q f = f$. Thus $f \in K$.

We want to show that $g \in K$. By passing to yet another subsequence of the (g_j) we can assure that either $m_j \to \infty$ as $j \to \infty$, or there is an $m \in \mathbb{N}$ such that $m_j = m$ for all $j \in \mathbb{N}$. If the first case occurs or if $m \geq q$ in the second case, $g_j = Q^*_{m_j} f_{\vec{k}_j}$ converges ω^* to f, hence $g = f \in K$. If on the other hand the second case applies and $m < q$, $g_j = Q^*_{m_j} f_{\vec{k}_j}$ converges ω^* to $Q^*_m f$. Since $f \in K$, $g = Q^*_m f \in K$. ∎

The major tool for the construction of the subarray (y^n_i) in Proposition 3.8 is the next lemma, which is a variation (and generalization) of a lemma employed by J. Elton in his thesis [**6**]. It allows us to "almost" achieve the condition in (21) for one row preceding the k_i's. Its versatility will allow repeated application to construct the array (y^n_i) row by row, carefully preserving properties obtained in previous steps.

Lemma 3.9. *Let $n_0 \in \mathbb{N}$ and let $(x^n_i)_{(i,n)\in T_{n_0}}$ be our bad $u\ell_p$-array indexed trapezoidally. Let $B \subseteq 2\, Ba\, X^*$, $C > 0$, $\varepsilon > 0$, $\delta > 0$, $N \in \mathcal{P}_\infty(\mathbb{N})$ and $n \geq n_0$ be given. Then there exists $L \in \mathcal{P}_\infty(N)$ such that if $(\ell_i)_{i=0}^q \subseteq L$ with $n \leq \ell_0 < \ell_1 < \ldots < \ell_q$ is given and if there exists an $f \in B$ with*

$$(22) \qquad \sum_{j=1}^q f(\alpha_j x^{n_0}_{\ell_j}) > C \quad \textit{for some} \quad (\alpha_j) \quad \textit{with} \quad \left(\sum_{j=1}^q |\alpha_j|^p\right)^{1/p} \leq \delta$$

then there exists a $g \in B$ with

$$(23) \qquad \begin{cases} \displaystyle\sum_{j=1}^q g(\beta_j x^{n_0}_{\ell_j}) > C \quad \textit{for some} \quad (\beta_j) \quad \textit{with} \quad \left(\sum_{j=1}^q |\beta_j|^p\right)^{1/p} \leq \delta \\ \textit{and} \quad \left|g(x^m_{\ell_0})\right| < \varepsilon \textit{ for } 1 \leq m \leq n\ . \end{cases}$$

Proof. We let

$$\mathcal{A}_q = \Big\{ I = (\ell_j)_{j=0}^\infty \in \mathcal{P}_\infty(N) \mid \ell_0 \geq n \text{ and if there is an } f \in B \text{ satisfying (22),}$$
$$\text{then there is a } g \in B \text{ satisfying (23)} \Big\}\ .$$

Let $\mathcal{A} = \bigcap_{q=1}^\infty \mathcal{A}_q$. Every set $\mathcal{A}_q$ (and hence $\mathcal{A}$) is a Ramsey set, since the conditions imposed on I involve only the first $(q+1)$ elements in I. By Ramsey theory (see *e.g.*, [**14**] for a discussion of Ramsey theory) we can thus find an $L \in \mathcal{P}_\infty(N)$ such that either $\mathcal{P}_\infty(L) \subseteq \mathcal{A}$ or $\mathcal{P}_\infty(L) \subseteq \mathcal{P}_\infty(N) \setminus \mathcal{A}$. In the first case the proof is finished; we show that the second Ramsey alternative leads to a contradiction.

We write $L = \{\ell_0, \ell_1, \ldots\}$. For fixed $q \in \mathbb{N}$ and $1 \leq r \leq q$ we let $L_r = \{\ell_r, \ell_{q+1}, , \ell_{q+2} \ldots\}$. By our assumption $L_r \notin \mathcal{A}$, thus $L_r \notin \mathcal{A}_{s_r}$ for some $s_r \in \mathbb{N}$. Consequently there exist (α^r_j) with $(\sum_{j=1}^{s_r} |\alpha^r_j|^p)^{1/p} \leq \delta$ and a function $f_r \in B$ such that

$$f_r\left(\sum_{j=1}^{s_r} \alpha^r_j x^{n_0}_{\ell_{q+j}}\right) > C$$

and whenever $g(\sum_{j=1}^{s_r} \beta_j x^{n_0}_{\ell_{q+j}}) > C$ for some function $g \in B$ and some (β_j) with $(\sum_{j=1}^{s_r} |\beta_j|^p \leq \delta$, then $\left|g(x^m_{\ell_r})\right| \geq \varepsilon$ for some $1 \leq m \leq n$.

Let r_0 be chosen such that $s_{r_0} = \min\{s_r \mid 1 \leq r \leq q\}$. We obtain for each $1 \leq r \leq q$

$$C < f_{r_0}\left(\sum_{j=1}^{s_{r_0}} \alpha^{r_0}_j x^{n_0}_{\ell_{q+j}}\right) = f_{r_0}\left(\sum_{j=1}^{s_r} \alpha^{r_0}_j x^{n_0}_{\ell_{q+j}}\right)\ ,$$

where we set $\alpha^{r_0}_j = 0$ for $s_{r_0} < j \leq s_r$. It follows that for each r we can find an $1 \leq m_r \leq n$ with $\left|f_{r_0}(x^{m_r}_{\ell_r})\right| \geq \varepsilon$.

Now we let q vary. Set $g_q = f_{r_0}$ and let g be a ω^*-clusterpoint of $\{g_q\}_{q\in\mathbb{N}}$. Due to our construction we can find for each $r \in \mathbb{N}$ a column $m'_r, 1 \leq m'_r \leq n$, with $|g(x^{m'_r}_{\ell_r})| \geq \varepsilon$. Thus one of the sequences $(x^m_{\ell_i})_{i=1}^\infty$, $1 \leq m \leq n$, is not weakly null, a contradiction. ∎

Proof of Proposition 3.8. In preparation for the diagonal procedure which follows we will introduce the following quantities and sets.

Let $\varepsilon = \min\{1, M_{n_0}/4\}$. Let (b_i^n) be the biorthogonal functionals associated with the bimonotone basic sequence $(x_i^n)_{(i,n)\in T_{n_0}}$ in its lexicographical order. For $(i,n) \in T_{n_0}$ we choose $(\varepsilon_i^n) > 0$ such that

$$\sum_{i=n_0}^{\infty} \sum_{n=1}^{i} \varepsilon_i^n \, \|b_i^n\| < \varepsilon \ . \tag{24}$$

We fix for each $(i,n) \in T_{n_0}$ a finite $\varepsilon_i^n/2$-net in $[-1,1]$ denoted by H_i^n. Let

$$B^1 := \{f \in 2\, Ba\, X^* \mid f(x_i^n) \in H_i^n \ \text{ for all } \ (i,n) \in T_{n_0}\}$$

Let us observe that by (24), whenever we can find a $g \in Ba\, X^*$, $(\alpha_i) \in Ba\, \ell_p$, and $(\ell_i)_{i=1}^{\infty}$, $\ell_1 > n_0$, with $g(\sum_{n=1}^{\infty} \alpha_i x_{\ell_i}^{n_0}) > M_{n_0}$, then there is an $f \in B^1$ with $f(\sum_{n=1}^{\infty} \alpha_i x_{\ell_i}^{n_0}) > \frac{3}{4} M_{n_0}$.

Next we choose $\varepsilon_m > 0$ for $m \geq n_0$ so that

$$\sum_{m=n_0}^{\infty} m\varepsilon_m \sup\{\|b_i^n\| \mid (i,n) \in T_{n_0}, n \leq m\} < \varepsilon \ . \tag{25}$$

The suprema above are finite, since each of the sequences $(x_i^n)_{i=1}^{\infty}$ was assumed to be semi-normalized.

For $m \geq n_0$ we let Γ_m be a finite ε_m-net in the interval $(0, M]$ and we let Δ_m be a set of positive reals such that the set $\{\delta^p : \delta \in \Delta_m\}$ is a (2^{-mp})-net for $[0,1]$, which contains 1. Furthermore we require that $\Delta_m \subseteq \Delta_{m+1}$ for all $m \geq n_0$.

We are ready to start the induction. Choose $C_1 \in \Gamma_{n_0}$ and $\delta_1 \in \Delta_{n_0}$ arbitrarily, apply Lemma 3.9 to $(B^1, \varepsilon_{n_0}, C_1, \delta_1, n_0, \mathbb{N})$ and obtain $L_1^1 \in \mathcal{P}_\infty(\mathbb{N})$. We pick $\delta_2 \in \Delta_{n_0}, \delta_1 \neq \delta_2$, and apply the lemma to $(B^1, \varepsilon_{n_0}, C_1, \delta_2, n_0, L_1^1)$, obtaining a new infinite subset L_2^1. We continue applying Lemma 3.9 successively until we have exhausted all combinations for which $(C, \delta) \in \Gamma_{n_0} \times \Delta_{n_0}$. If L_1 is the last infinite subset of $\mathbb{N}$ thus obtained, we let $\ell_1 = \min L_1$. This defines the first row of the (trapezoidal) subarray (y_i^n): $y_{n_0}^j = x_{\ell_1}^j$ for $1 \leq j \leq n_0$. We set $F(n_0, j) = H_{\ell_1}^j$ for $1 \leq j \leq n_0$.

For the second step of the induction we first partition the set B^1 into finitely many sets as follows: For $\vec{t} = (t_1, \ldots, t_{n_0}) \in \prod_{j=1}^{n_0} F(n_0, j)$ we let

$$B_{\vec{t}}^2 = \left\{f \in B^1 \mid f(y_{n_0}^j) = t_j \ \text{ for all } \ 1 \leq j \leq n_0\right\} \ .$$

Similarly to what we did in the first step we apply Lemma 3.9 successively to $(B_{\vec{t}}^2, \varepsilon_{n_0+1}, C, \delta,\ n_0+1, \cdot)$ beginning with the infinite set L_1, until we have exhausted all combinations $(\vec{t}, C, \delta) \in \prod_{j=1}^{n_0}(F(n_0, j) \times \Gamma_{n_0+1} \times \Delta_{n_0+1})$. Let L_2 denote the last sequence thus obtained. We choose as ℓ_2 an element in L_2 with $\ell_2 > \ell_1$. This defines the second row of the subarray: $y_{n_0+1}^j = x_{\ell_2}^j$ for $1 \leq j \leq n_0+1$. We set $F(n_0+1, j) = H_{\ell_2}^j$ for $1 \leq j \leq n_0 + 1$.

For the general induction step let us assume that $\ell_1 < \ell_2 < \ldots < \ell_m$ and L_m have been found in the way now described. This defines the first m rows of $(y_i^n)_{(i,n)\in T_{n_0}}$. We set $F(m', j) = H_{\ell_m}^j$ for $1 \leq j \leq m'$, where $m' = n_0 + m - 1$. We partition B^1 — as before — into finitely many sets: for $\vec{t} = (t_i^n) \in \prod\{F(i,n) \mid (i,n) \in T_{n_0}$ and $i \leq m'\}$ we let

$$B_{\vec{t}}^{m+1} = \left\{f \in B^1 \mid f(y_i^n) = t_i^n \ \text{ for all } \ (i,n) \in T_{n_0} \ \text{ with } \ i \leq m'\right\} \ .$$

Now we apply Lemma 3.9, starting with the sequence L_m, successively to $(B_{\vec{t}}^{m+1}, \varepsilon_{m'+1}, C, \delta,\ m'+1, \cdot)$ where $(\vec{t}, C, \delta)$ ranges over all possible combinations in $\prod(F(i,n) \times \Gamma_{m'+1} \times \Delta_{m'+1})$. If L_{m+1} is the last sequence we obtain, we choose $\ell_{m+1} \in L_{m+1}$ with $\ell_{m+1} > \ell_m$. The induction is complete.

From now on we will identify the functionals $f \in B^1$ with their restrictions to $Y = [y_i^n]_{(i,n)\in T_{n_0}}$.

The subarray $(y_i^n)_{(i,n)\in T_{n_0}}$ of $(x_i^n)_{(i,n)\in T_{n_0}}$ has been chosen such that the following holds:

Lemma 3.10. *Let $n_0 \leq n < k_1 < k_2 < \ldots < k_r$ be given. If there exists an $f \in B^1$ such that $\sum_{i=1}^r f(\alpha_i y_{k_i}^{n_0}) > C$ for some (α_i) with $(\sum_{i=1}^r |\alpha_i|^p)^{1/p} \leq \delta$, $\delta \in \Delta_n$, $C \in \Gamma_n$, then there exists $g \in B^1$ with*

$$\sum_{i=1}^r g(\beta_i y_{k_i}^{n_0}) > C \quad \text{for some} \quad (\beta_i) \quad \text{with} \quad \left(\sum_{i=1}^r |\beta_i|^p\right)^{1/p} \leq \delta \tag{26}$$

$$g(y_i^m) = f(y_i^m) \in F(i,m) \quad \text{for all} \quad (i,m) \in T_{n_0} \quad \text{with} \quad i < n \tag{27}$$

$$|g(y_n^m)| < \varepsilon_n \quad \text{for all} \quad 1 \leq m \leq n \ . \tag{28}$$

We claim that the proof of Proposition 3.8 is complete, once we prove

Lemma 3.11. *Let $n_0 < k_1 < \ldots < k_q$ be given. If $\left\|\sum_{i=1}^q \alpha_i y_{k_i}^{n_0}\right\| > M_{n_0}$ for some (α_i) with $(\sum_{i=1}^q |\alpha_i|^p)^{1/p} \leq 1$, then there exists an $h \in B^1$ such that*

$$\sum_{i=1}^q h(\gamma_i y_{k_i}^{n_0}) > M_{n_0}/4 \quad \text{for some} \quad (\gamma_i) \quad \text{with} \quad \left(\sum_{i=1}^q |\gamma_i|^p\right)^{1/p} \leq 1 \tag{29}$$

$$h(y_i^n) = 0 \text{ for all } n \geq n_0, \quad \text{if} \quad i > k_q \tag{30}$$

$$|h(y_i^n)| < \varepsilon_i \text{ for all } n \geq n_0, \quad \text{if} \quad i \notin \{k_1, k_2, \ldots, k_q\} \ . \tag{31}$$

Completion of the proof of Proposition 3.8. Indeed, by perturbing h, we can obtain an $f \in Y^*$ such that for $(i,n) \in T_{n_0}$, $f(y_i^n) = h(y_i^n)$ if $i \in \{k_1, k_2, \ldots, k_q\}$ and $f(y_i^n) = 0$ otherwise. Thus (21) is satisfied. Moreover it follows that $\sum_{i=1}^q f(\gamma_i y_{k_i}^{n_0}) = \sum_{i=1}^q h(\gamma_i y_{k_i}^{n_0}) > M_{n_0}/4$ and $f(y_i^n) \in F(i,n)$ for all $(i,n) \in T_{n_0}$. Thus (19) and (20) hold. By using (25) we can estimate $\|f\|$ as follows:

$$\begin{aligned} \|f - h\| &\leq \sum_{i=n_0}^{k_q} \varepsilon_i \sum_{n=1}^{i} \left\|b_{\ell_i}^n\right\| \\ &\leq \sum_{m=n_0}^{\infty} m \cdot \varepsilon_m \cdot \sup\{\|b_i^n\| \mid (i,n) \in T_{n_0}, n \leq m\} \\ &< \varepsilon \leq 1 \ . \end{aligned}$$

Since $\|h\| \leq 2$, we have $\|f\| \leq 3$. This ends the proof of Proposition 3.8. ∎

Proof of Lemma 3.11. If $\|\sum_{i=1}^q \alpha_i y_{k_i}^{n_0}\| > M_{n_0}$ for some $q \in \mathbb{N}$, $n_0 \leq k_1 < \ldots < k_q$ and some (α_i) with $(\sum_{i=1}^q |\alpha_i|^p)^{1/p} \leq 1$, we can find by our earlier observation a function $g \in B^1$ with $g(\sum_{i=1}^q \alpha_i y_{k_i}^{n_0}) > \frac{3}{4} M_{n_0}$.

We will apply Lemma 3.10 (at most) $(k_q - n_0 + 1)$ times beginning with the function g and the row $n = n_0$. We choose $C_{n_0} \in \Gamma_{n_0}$ such that $0 < \frac{3}{4} M_{n_0} - C_{n_0} < \varepsilon_{n_0}$.

If $n_0 = k_1$, we set $h_{n_0} = g$ and $\alpha_{i,n_0} = \alpha_i$ for $1 \leq i \leq q$. We set $\gamma_1 = \alpha_{1,n_0}$, if $h_{n_0}(\alpha_{1,n_0} y_{n_0}^{n_0}) \geq 0$, and $\gamma_1 = -\alpha_{1,n_0}$ otherwise. We let $\beta_{n_0} = h_{n_0}(\gamma_1 y_{n_0}^{n_0})$ and choose a $\delta_1 \in \Delta_{n_0}$ such that $\delta_1^p - 2^{-p} \leq \sum_{j=2}^q |\alpha_{j,n_0}|^p \leq \delta_1^p$.

If on the other hand $n_0 < k_1$, we apply Lemma 3.10 to g, $n_0 = n < k_1 < \ldots < k_q$, $\delta = 1$ $(\in \Delta_{n_0})$ and $C = C_{n_0} \in \Gamma_{n_0}$. Note that $C < \frac{3}{4} M_{n_0}$ and that therefore the premise of the lemma is indeed fulfilled. The application yields a new functional $h_{n_0} \in B^1$ and a new $(\alpha_{i,n_0})_{i=1}^q \in Ba\,\ell_p^q$ with $\sum_{i=1}^q h_{n_0}(\alpha_{i,n_0} y_{k_i}^{n_0}) > C_{n_0}$ and $|h_{n_0}(y_{n_0}^m)| < \varepsilon_{n_0}$ for $1 \leq m \leq n_0$. We let $\beta_{n_0} = 0$ and $\delta_1 = 1$.

Let us now assume that $s \geq n_0$ and we have thus far constructed $h_s \in B^1$, $\alpha_{i,s}$ for $i = 1, \ldots, q$ and for $n_0 \leq r \leq s$, scalars $C_r \in \Gamma_r$, $\delta_r \in \Delta_r$ and $\beta_r \geq 0$, and furthermore for each $1 \leq k_i \leq s$ we have chosen $\gamma_i \in \mathbb{R}$ such that the following conditions hold:

$$0 < (C_{r-1} - \beta_{r-1}) - C_r < \varepsilon_r \quad \text{for all} \quad n_0 \leq r \leq s \tag{32}$$

$$\left(\sum_{\{i|k_i > r\}} |\alpha_{i,r}|^p \right)^{1/p} \leq \delta_j \quad \text{for all} \quad n_0 \leq r \leq s \quad \text{with} \quad k_j \leq r < k_{j+1} \tag{33}$$

$$\delta_j^p - 2^{-jp} \leq \sum_{i=j+1}^{q} |\alpha_{i,k_j}|^p \quad \text{for all} \quad n_0 \leq k_j \leq s \tag{34}$$

$$\sum_{\{i|k_i \geq s\}} h_s(\alpha_{i,s} y_{k_i}^{n_0}) > C_s \tag{35}$$

$$\beta_r = h_s(\gamma_i y_{k_i}^{n_0}) \quad \text{if} \quad r = k_i \quad \text{for some} \quad 1 \leq i \leq q \ , \quad \text{and} \quad \beta_r = 0 \quad \text{otherwise} \tag{36}$$

$$|h_s(y_r^m)| < \varepsilon_r \quad \text{for all} \quad 1 \leq m \leq r \leq s \quad \text{with} \quad r \notin \{k_1, k_2, \ldots, k_q\} \ . \tag{37}$$

Let us observe that all these conditions are satisfied for $s = n_0$, if we let $C_{n_0-1} = \frac{3}{4} M_{n_0}$, $\beta_{n_0-1} = 0$, $k_0 = n_0$ and $\delta_0 = 1$. (We also set $\alpha_{i,n_0-1} = \alpha_i$ for $1 \leq i \leq q$.)

Next we choose $C_{s+1} \in \Gamma_{s+1}$ with $0 < (C_s - \beta_s) - C_{s+1} < \varepsilon_{s+1}$ to satisfy (32) for $s+1$. Note that $C_{s+1} < C_s$. (If $C_s - \beta_s < \varepsilon_{s+1}$, we quit the procedure and set $h = Q_s^* h_s$. Estimates below will show that h satisfies the conclusion of Lemma 3.11. We set $\gamma_j = 0$ for $k_j > s$.)

If $s+1 = k_j$ for some $1 \leq j \leq q$ we set $h_{s+1} = h_s$ and $\alpha_{i,s+1} = \alpha_{i,s}$ for $1 \leq i \leq q$. We set $\gamma_j = \pm\alpha_{j,s+1}$ so that $\beta_{s+1} := h_{s+1}(\gamma_j y_{k_j}^{n_0}) \geq 0$ and choose $\delta_j \in \Delta_{s+1}$ so that (33) and (34) are satisfied for $s+1$. (36) is satisfied for $s+1$ by our construction; in (37) no new condition is imposed, so it remains to check (35). Indeed we have

$$\begin{aligned}
\sum_{\{i|k_i \geq s+1\}} h_{s+1}(\alpha_{i,s+1} y_{k_i}^{n_0}) &= \sum_{\{i|k_i \geq s+1\}} h_s(\alpha_{i,s} y_{k_i}^{n_0}) \\
&\geq \sum_{\{i|k_i \geq s\}} h_s(\alpha_{i,s} y_{k_i}^{n_0}) - \beta_s \quad \text{by (36)} \\
&> C_s - \beta_s > C_{s+1} \quad \text{using (35) for } r = s \text{ and (32) for } r = s+1 \ .
\end{aligned}$$

If $s+1 \notin \{k_1, k_2, \ldots, k_q\}$, say $k_{j-1} < s+1 < k_j$ for some j, we apply Lemma 3.10 to h_s, $n = s+1 < k_j < \ldots < k_q$, C_{s+1}, δ_{j-1} and $(\alpha_{i,s})$, $j \leq i \leq q$. Let us check that the lemma applies for these parameters: Clearly $C_{s+1} \in \Gamma_{s+1}$ and $\delta_{j-1} \in \Delta_s \subseteq \Delta_{s+1}$. Moreover

$$\left(\sum_{i=j}^{q} |\alpha_{i,s}|^p \right)^{1/p} = \left(\sum_{\{i|k_i > s\}} |\alpha_{i,s}|^p \right)^{1/p} \leq \delta_{j-1} \quad \text{by (33).}$$

Finally, as above, we obtain

$$\sum_{i=j}^{q} h_s(\alpha_{i,s} y_{k_i}^{n_0}) = \sum_{\{i|k_i \geq s+1\}} h_s(\alpha_{i,s} y_{k_i}^{n_0}) > C_{s+1} \ .$$

The application of Lemma 3.10 yields a new functional $h_{s+1} \in B^1$, and new $(\alpha_{i,s+1})$ for $j \le i \le q$ with $(\sum_{i=j}^q |\alpha_{i,s+1}|^p)^{1/p} \le \delta_{j-1}$, $\sum_{i=j}^q h_{s+1}(\alpha_{i,s+1} y_{k_i}^{n_0}) > C_{s+1}$ and $\left|h_{s+1}(y_{s+1}^m)\right| < \varepsilon_{s+1}$ for all $1 \le m \le s+1$; hence (37) is satisfied for $r = s+1$. Since h_{s+1} preserves the values of h_s on the rows prior to the $(s+1)$st row, h_{s+1} satisfies (37) also for $1 \le r \le s$. We set $\beta_{s+1} = 0$. By our construction (33), (35) and (36) are satisfied for $s+1$; (34) does not impose a new condition.

Unless we stopped the construction earlier, we quit after we have obtained h_{k_q} and let $h = Q_{k_q}^* h_{k_q}$. Observing that the conclusions (30) and (31) in Lemma 3.11 hold, it remains to check (29). If h is defined to be $h = Q_{k_q}^* h_{k_q}$ we obtain

$$\begin{aligned}
\sum_{i=1}^q h(\gamma_i y_{k_i}^{n_0}) &= \sum_{i=1}^q \beta_{k_i} \qquad \text{by (36)} \\
&= \beta_{k_q} + \sum_{r=n_0}^{k_q} \beta_{r-1} \\
&\ge \beta_{k_q} + \sum_{r=n_0}^{k_q} (C_{r-1} - C_r - \varepsilon_r) \\
&\ge (\beta_{k_q} - C_{k_q}) + \tfrac{3}{4} M_{n_0} - \sum_{r=n_0}^{k_q} \varepsilon_r \\
&\ge \tfrac{3}{4} M_{n_0} - \varepsilon \;\ge\; \tfrac{1}{2} M_{n_0}\ .
\end{aligned}$$

(Note that $\beta_{k_q} - C_{k_q} \ge 0$ by (35) and (36).)
A similar estimate holds if $h = Q_s^* h_s$ for some $s < k_q$:

$$\begin{aligned}
\sum_{i=1}^q h(\gamma_i y_{k_i}^{n_0}) &\ge \beta_s + \sum_{r=n_0}^{s} (C_{r-1} - C_r - \varepsilon_r) \\
&\ge (\beta_s - C_s) + \left(\tfrac{3}{4} M_{n_0} - \sum_{r=n_0}^{s} \varepsilon_r\right) \;\ge\; \tfrac{3}{4} M_{n_0} - \varepsilon \;\ge\; \tfrac{1}{2} M_{n_0}\ .
\end{aligned}$$

(This time $\beta_s - C_s \ge -\varepsilon_{s+1}$ by our stopping condition.)

The proof is complete, once we show that $(\sum_{j=1}^q |\gamma_j|^p)^{1/p} \le 2$. Indeed, if $h = Q_{k_q}^* h_{k_q}$, we have for $1 \le j < q$,

$$\begin{aligned}
|\gamma_j|^p = |\alpha_{j,k_j-1}|^p &\le \sum_{i=j}^q |\alpha_{i,k_j-1}|^p - \sum_{i=j+1}^q |\alpha_{i,k_j}|^p \\
&\le \delta_{j-1}^p - (\delta_j^p - 2^{-jp}) \;\text{ by (33) and (34);} \\
|\gamma_q|^p = |\alpha_{q,k_q-1}|^p &\le \delta_{q-1}^p \;\text{ by (33).}
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\sum_{j=1}^q |\gamma_j|^p &\le \sum_{j=1}^{q-1} (\delta_{j-1}^p - \delta_j^p) + \delta_{q-1}^p + \sum_{j=1}^{q-1} 2^{-jp} \\
&\le 1 + \sum_{j=1}^{q-1} 2^{-jp} \le 2\ .
\end{aligned}$$

We obtain the same estimate, if we stopped the procedure before reaching the row k_q. ∎

4. Proofs of the corollaries

Proposition 3.8 can be phrased for a single weakly null sequence as follows:

Corollary 3. *Let (x_n) be a semi-normalized weakly null sequence in a Banach space X. Assume none of its subsequences has a C-upper ℓ_p-estimate. Then there exists a subsequence $(y_n) \subseteq (x_n)$, such that for all $(m_n) \in \mathcal{P}_\infty(\mathbb{N})$ there are a functional $f \in Ba([y_n]^*)$, an $\ell \in \mathbb{N}$ and $(\alpha_i)_{i=1}^\ell \in Ba\,\ell_p$ such that*

$$f\left(\sum_{i=1}^{\ell} \alpha_i y_{m_i}\right) > C/12$$

$$\text{and } f(y_i) = 0, \text{ if } i \notin \{m_1, \ldots, m_\ell\}\ .$$

C. Schumacher [**16**] uses this corollary to deduce the following weak-Cauchy criterion for property (S_p), which is a generalization of an analogous result in the c_0-case in [**11**]:

Proposition 4.1. *Let X be a Banach space. The following are equivalent:*

(i) X has property (S_p).

(ii) Every weak Cauchy-sequence in X has a subsequence (y_n) such that, for some constant $C < \infty$, all subsequences (y'_n) of (y_n) satisfy

$$\left\|\sum_{n=1}^{\infty} a_n (y'_n - y'_{n-1})\right\| < C \quad \text{for all} \quad (a_n) \in Ba\,\ell_p\ .$$

(Here $y_0 = 0$.)

Corollary 1 is an easy consequence of this proposition:

Proof of Corollary 1. Let (z_n) be a weak Cauchy-sequence in $Ba(X/Y)$ and let $q : X \to X/Y$ denote the quotient map. By a result of R.H. Lohman [**13**] we can find a weak Cauchy-sequence (x_n) in X such that its image under q is some subsequence of (z_n). Since X has property (S_p), (x_n) satisfies the conclusion of (ii) of Proposition 4.1; so does its image under q, which is still a subsequence of (z_n). ∎

Next we present the proof of the second corollary:

Proof of Corollary 2. We will only prove the second statement. The proof of the first statement is quite similar and will be left to the reader. By Theorem 1 we can find constants C_p and C^q such that every weakly null sequence in $Ba\,X$ (resp. in $Ba\,X^*$) admits a subsequence with a C_p-upper ℓ_p-estimate (resp. C^q-upper ℓ_q-estimate). Let (f_n) be a normalized weakly null sequence in X^*. By passing to a subsequence we may assume that (f_n) has a C^q-upper ℓ_q-estimate. We choose a separable subspace Y of X which isometrically norms all the f_n's and denote by g_n the restriction of f_n to Y. Using a result due to W.B. Johnson and H.P. Rosenthal [**10**], we can find a basic sequence $(x_n) \subseteq Y$ with $\|x_n\| \le 3$ for all n, such that a subsequence of (g_n), which we still denote by (g_n), is biorthogonal to (x_n). Since ℓ_1 does not embed into Y, we can assume that (x_n) is a weak Cauchy sequence. Furthermore, since Y has property (S_p) we may assume, by passing to a subsequence of (x_n), that $y_n := x_{2n} - x_{2n-1}$ has a $6C_p$-$u\ell_p$-estimate.

We claim that (f_{2n}) has a lower ℓ_q-estimate. Indeed, let (b_n) be given with $\left(\sum_{n=1}^\infty |b_n|^q\right)^{1/q} = 1$. Choose (a_n) with $\left(\sum_{n=1}^\infty |a_n|^p\right)^{1/p} = 1$ and $\sum_{n=1}^\infty a_n b_n = 1$. We obtain the following estimate:

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} b_n f_{2n}\right\| &\ge \left(\left\|\sum_{m=1}^{\infty} a_m y_m\right\|\right)^{-1} \cdot \left|\left(\sum_{n=1}^{\infty} b_n f_{2n}\right) \cdot \left(\sum_{m=1}^{\infty} a_m y_m\right)\right| \\ &\ge (6C_p)^{-1} \cdot \left|\sum_{n=1}^{\infty} a_n b_n\right| \quad = \quad (6C_p)^{-1}\ .\end{aligned}$$

The projection $Q : X^* \longrightarrow [f_{2n}]$ is defined by $Qf = \sum_{n=1}^{\infty} f(y_n) f_{2n}$. Note that f is only applied to the $u\ell_p$-sequence (y_n) in Y; thus Q is well defined. It is easy to check that Q is the identity map on $[f_{2n}]$ and that $\|Q\| \leq 6C_pC^q$. ∎

J. Elton obtained in [**6**] the following characterization: *Let (x_n) be a semi-normalized weakly null sequence in a Banach space without a subsequence equivalent to the unit vector basis of c_0. Then (x_n) has a subsequence (y_n) such that $\|\sum_{n=1}^k a_n y_n\| \to \infty$ as $k \to \infty$, whenever $(a_n) \notin c_0$.*

We conclude by showing that an analogous result fails in a strong way for the ℓ_p-case:

Proposition 4.2. *Let $1 < p < \infty$. There is a Banach space X with a 1-symmetric basis (e_n) such that the following properties hold:*

(i) (e_n) does not have an upper ℓ_p-estimate.

(ii) There is a sequence $(\alpha_n) \notin \ell_p$ such that $\sum_{n=1}^{\infty} a_n e_n$ converges.

Proof. Choose $1 < p_0 < p < p_1$. It is easy to construct a concave increasing function $\lambda : \mathbb{N} \to \mathbb{R}^+$ with $\lambda(1) = 1$, such that both $B := \{n : \lambda(n) \geq n^{1/p_0}\}$ and $L := \{n : \lambda_n \leq n^{1/p_1}\}$ are infinite subsets of $\mathbb{N}$. Let X be the Lorentz sequence space $d(w, 1)$, where $w_1 = \lambda(1)$ and $w_n = \lambda(n) - \lambda(n-1)$ for $n > 1$. X has a 1-symmetric basis (e_n) with the property $\lambda(n) = \|\sum_{i=1}^n e_i\|$ (see [**12**, I, p.120]). Since $\#B = \infty$, (e_n) does not have an upper ℓ_p-estimate. To see (ii) we proceed as follows: we choose an increasing sequence $(\ell_k) \subset L$ with

$$\ell_k^{1-(p/p_1)} > k^{2p} \ . \tag{37}$$

Set $m_0 = 0$, $m_j = \sum_{k=1}^j \ell_k$ and $F_j = (m_{j-1}, m_j]$ for $j \geq 1$. We consider $\sum_{n=1}^{\infty} a_n e_n$, where

$$a_n = j^{-2} \ell_j^{-1/p_1} \ , \quad \text{if} \ \ n \in F_j \ .$$

Since $\ell_j \in L$, $\ell_j^{-1/p_1} \|\sum_{n \in F_j} e_n\| \leq 1$, and hence

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\| = \left\| \sum_{j=1}^{\infty} j^{-2} \ell_j^{-1/p_1} \left(\sum_{n \in F_j} e_n \right) \right\| < \sum_{j=1}^{\infty} j^{-2} < \infty \ .$$

On the other hand, for $N \in \mathbb{N}$,

$$\sum_{n=1}^{m_N} |a_n|^p = \sum_{j=1}^{N} \ell_j (j^{-2} \ell_j^{-1/p_1})^p = \sum_{j=1}^{N} \ell_j^{1-(p/p_1)} j^{-2p} \geq N \quad \text{by (37)} \ .$$

Thus $(a_n) \notin \ell_p$. ∎

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