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Orlicz sequence spaces of Banach-Saks type

By

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We show that in Orlicz sequence spaces of Banach-Saks type p, weakly null sequences admit of p-Hilbertian subsequences.

W. B. Johnson introduced the following notion in [3].

Definition 1. Let $1 < p \leqq \infty$. A Banach space has *Banach-Saks type p* (property (BS_p), for short), if every weakly null sequence has a subsequence (x_k) so that for some $C < \infty$

$$\left\| \sum_{k=1}^{n} x_k \right\| \leqq C n^{1/p} \quad \text{for all } n \in \mathbb{N}.$$

(Here, $n^{1/\infty} := 1$ for all $n \in \mathbb{N}$.)

The following stronger property was introduced in [4]:

Definition 2. Let $1 < p \leqq \infty$. A Banach space has *property* (S_p), if every weakly null sequence has a subsequence (x_k) so that for some $C < \infty$

$$\left\| \sum_{k=1}^{n} a_k x_k \right\| \leqq C \quad \text{for all } n \in \mathbb{N} \text{ and for all scalars } (a_k) \text{ with } \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \leqq 1.$$

(Here, $(\sum |a_k|^\infty)^{1/\infty} := \max |a_k|$.)

A sequence (x_k), which is dominated by the unit vector basis of ℓ_p, i.e., which satisfies the estimate in Definition 2, is called *p-Hilbertian*.

It follows from Elton's c_0-theorem ([1], see also [6], Corollary 4.4) that property (BS_∞) implies property (S_∞). Moreover, both properties are equivalent to the hereditary Dunford-Pettis property. For $1 < p < \infty$, however, the two properties are not equivalent: the Lorentz sequence space $d(\mathbf{w}, 1)$, where $\mathbf{w} = (w_i)$ satisfies $\sum_{i=1}^{n} w_i = n^{1/p}$ for all $n \in \mathbb{N}$, has property (BS_p), while failing property (S_p). S. A. Rakov ([7], Theorem 3) proved that property (BS_p) implies property $(S_{p-\varepsilon})$ for all $\varepsilon > 0$.

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We can now state our result:

Theorem 3. *Let $1 < p < \infty$ and let h_M be an Orlicz sequence space not containing ℓ_1. Then the following properties are equivalent:*
(a) *h_M has property* (BS_p).
(b) *The Orlicz function M satisfies:*

$$\sup_{0<s,t\leqq 1} \frac{M(st)}{M(s)\,t^p} < \infty.$$

(c) *h_M has property* (S_p).

Thus, in this context, Orlicz sequence spaces show the same behavior as the classical ℓ_p spaces.

An *Orlicz function* $M\colon [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing and convex function satisfying $M(0) = 0$ and $\lim\limits_{t\to\infty} M(t) = \infty$. We will also assume that $M(t)$ is non-degenerate ($M(t) > 0$ for all $t > 0$). The *Orlicz sequence space h_M, with Orlicz function M,* is the Banach space consisting of all sequences (a_k) of scalars so that

$$\sum_{k=1}^{\infty} M\left(\frac{|a_k|}{\varrho}\right) < \infty \quad \text{for all } \varrho > 0,$$

equipped with the norm

$$\|(a_k)\|_{h_M} = \inf\left\{\varrho > 0 \,\middle|\, \sum_{k=1}^{\infty} M\left(\frac{|a_k|}{\varrho}\right) \leqq 1\right\}.$$

For basic facts about Orlicz sequence spaces we refer the reader to [5]. Let us note that the expression in (b) is related to the question, for what values of p the space ℓ_p embeds into h_M (see [5], Theorem 4.a.9):

$$\begin{aligned}\alpha_M := \sup\left\{p \,\middle|\, \sup_{0<s,t\leqq 1} \frac{M(st)}{M(s)\,t^p} < \infty\right\}\\ = \min\{p \mid \ell_p \text{ is isomorphic to a subspace of } h_M\}.\end{aligned}$$

Our result improves on a result by Rakov ([7], Theorem 5), who showed that

$$\alpha_M = \sup\{p \mid h_M \text{ has property } (\mathrm{BS}_p)\} = \sup\{p \mid h_M \text{ has property } (\mathrm{S}_p)\}.$$

Let us remark that the Orlicz sequence space with an Orlicz function $M(t)$, satisfying $M(t) = -t^p \log t$ for small t, has property $(\mathrm{BS}_{p-\varepsilon})$ for all $\varepsilon > 0$, while it fails property (BS_p).

Proof of Theorem 3. (c) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (b): For $k \in \mathbb{N}$ we let α_k be a solution of $k \cdot M(\alpha_k) = 1$. (α_k) is a decreasing sequence of positive reals with $\lim\limits_{k\to\infty} \alpha_k = 0$. α_k has been chosen so that $b^{(k)} = \sum\limits_{i=1}^{k} \alpha_k e_i$ has norm 1 in h_M. (Here (e_i) denotes the unit vector basis in h_M.)

We consider the sequence $b_m^{(k)} = \sum_{i=1}^{k} \alpha_k e_{mk+i}$ for $m \in \mathbb{N}$. Since ℓ_1 does not embed into h_M, the sequence $(b_m^{(k)})_{m \in \mathbb{N}}$ is weakly null for all $k \in \mathbb{N}$. It follows from the symmetry of (e_i) and a result by Rakov ([7], see [2], Proposition 3.2), that there is a constant $C < \infty$, *independent of k*, so that for all $k \in \mathbb{N}$

$$\left\| \sum_{m=1}^{n} b_m^{(k)} \right\| \leqq C n^{1/p} \quad \text{for all } n \in \mathbb{N}.$$

Consequently,

$$n \cdot k \cdot M\left(\frac{\alpha_k}{C n^{1/p}}\right) \leqq 1.$$

Let now $0 < s, t \leqq 1$ be given. We choose $k \in \mathbb{N}$ so that $\alpha_{k+1} < s \leqq \alpha_k$, and $n \in \mathbb{N}$ so that $1/C(n+1)^{1/p} < t \leqq 1/C n^{1/p}$. (We may assume that $s \leqq \alpha_1$ and that $t \leqq 1/C$, since it suffices to establish (b) for small s and t (see [5], Proposition 4.a.5.).)

We obtain the desired estimate as follows:

$$\begin{aligned} \frac{M(st)}{M(s)\,t^p} &\leqq \frac{M\left(\dfrac{\alpha_k}{C n^{1/p}}\right)}{M(\alpha_{k+1})\left[\dfrac{1}{C(n+1)^{1/p}}\right]^p} \\ &\leqq C^p \frac{n+1}{n} \frac{k+1}{k} \\ &\leqq 4 C^p. \end{aligned}$$

(b) $\Rightarrow$ (c): Let (x_i) be a weakly null sequence in h_M satisfying $\|x_i\| \leqq 1$ for all $i \in \mathbb{N}$. By applying a standard perturbation argument we can assume that $x_i = \sum_{j \in F_i} a_j e_j$ for some finite integer blocks $F_1 < F_2 < \ldots$. Since $\|x_i\|_{h_M} \leqq 1$, $\sum_{j \in F_i} M(|a_j|) \leqq 1$.

By our hypothesis there is a constant C so that

$$\sup_{0 < s, t \leqq 1} \frac{M(st)}{M(s)\,t^p} \leqq C^p.$$

Let (β_i) be given with $\sum |\beta_i|^p \leqq 1$.

We show that $\left\| \sum_{i=1}^{\infty} \beta_i x_i \right\| \leqq C$. Indeed:

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j \in F_i} M\left(\frac{|\beta_i a_j|}{C}\right) &\leqq \sum_{i=1}^{\infty} \sum_{j \in F_i} C^p M(|a_j|) \frac{|\beta_i|^p}{C^p} \\ &\leqq \sum_{i=1}^{\infty} |\beta_i|^p \cdot \sum_{j \in F_i} M(|a_j|) \\ &\leqq \sum_{i=1}^{\infty} |\beta_i|^p \leqq 1. \end{aligned}$$

Remarks. 1. If ℓ_1 embeds into h_M, then $\alpha_M = 1$. Rakov observed in ([8], Remark 6), that in this case either h_M satisfies property (BS_∞), and thus property (S_∞), or h_M fails property (BS_p), and thus property (S_p), for all $p > 1$. Consequently, property (BS_p) and property (S_p) are equivalent for *all* Orlicz sequence spaces h_M.

2. We do not know of an example of a Banach space, not containing ℓ_1, which has property (BS_p) and fails property (S_p).

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