# How Do Calculators Calculate?

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We give an introduction to the

**CORDIC method**

used my most handheld calculators (such as the ones by Texas Instruments and Hewlett-Packard) to approximate the standard transcendental functions.
The CORDIC algorithm does not use Calculus based methods such as polynomial or rational function approximation.

The CORDIC (= **CO**ordinate **R**otation **DI**gital **C**omputer) algorithm was developed by

**Jack E. Volder**

in 1959.
His objective was to build a real-time navigational computer for use on aircrafts, so he was primarily interested in computing trigonometric functions.
Subsequently, the CORDIC scheme was extended by

**J. Walther**

in 1971 to other transcendental functions.

Hand-held calculators do not convert numbers to base 2.
They use a binary-coded decimal (BCD) system instead.
Calculators can only perform four operations inexpensively:

1. **Addition** and **Subtraction**
2. **Storing** in memory and **Retrieving** from memory
3. **Digit shift** (multiplication/division by the base)
4. **Comparisons**

The CORDIC Algorithm is a unified computational scheme to perform

1. multiplication and division
2. computations of the trigonometric functions
$$\sin, \cos, \arctan$$
3. computations of the hyperbolic trigonometric functions
$$\sinh, \cosh, \operatorname{arctanh}$$
4. and consequently can also compute the exponential function, the natural logarithm and the square root

## The CORDIC Algorithm (Binary Version)

$$x_{k+1} = x_k - m\delta_k y_k 2^{-k} \quad (1)$$

$$y_{k+1} = y_k + \delta_k x_k 2^{-k} \quad (2)$$

$$z_{k+1} = z_k - \delta_k \varepsilon_k \quad (3)$$

$$\delta_k = \pm 1 \text{ (depending on } y_k \text{ or } z_k) \quad (4)$$

$$m = 0, \pm 1 \quad (5)$$

The $\varepsilon_k$'s are prestored constants.

## A Warm-Up Example: Multiplication

To compute the product of $a$ and $b$ for $|a| \leq 2$ and $|b| \leq 2$ we let

$$m = 0,\ \varepsilon_k = 2^{-k}.$$

The scheme becomes

$$\begin{aligned} x_{k+1} &= x_k \quad (= x_0) && (6)\\ y_{k+1} &= y_k + \delta_k x_k 2^{-k} && (7)\\ z_{k+1} &= z_k - \delta_k 2^{-k} && (8) \end{aligned}$$

$$\delta_k = \operatorname{sgn}(z_k) \quad (9)$$

$$\begin{aligned} x_0 &= a && (10)\\ y_0 &= 0 && (11)\\ z_0 &= b && (12) \end{aligned}$$

From the previous examples we suspect that $z_n \to 0$ and that $y_n \to ab$. Let's check:

**Theorem**

$|z_{n+1}| \leq 2^{-n}$

**Proof by induction:** By the restriction placed on $b$

$$|z_0| = |b| \leq 2^{-(-1)} = 2.$$

Now suppose $|z_n| \leq 2^{-(n-1)} = 2^{-n+1}$. By definition $z_{n+1} = z_n - \operatorname{sgn}(z_n)2^{-n}$. There are two cases to consider:
If $0 \leq z_n \leq 2^{-n+1}$, then $\operatorname{sgn}(z_n) = 1$, and consequently $-2^{-n} \leq z_{n+1} \leq 2^{-n}$.
If on the other hand $-2^{-n+1} \leq z_n < 0$, then $\operatorname{sgn}(z_n) = -1$, and consequently $-2^{-n} \leq z_{n+1} \leq 2^{-n}$.

### Theorem

$|y_{n+1} - ab| \leq 2^{-n+1}$

**Proof:** Note that the iterative formulas imply:

$$z_{n+1} = z_0 - \sum_{k=0}^{n} \delta_k 2^{-k},$$

and

$$y_{n+1} = y_0 + \left( \sum_{k=0}^{n} \delta_k 2^{-k} \right) x_0.$$

Consequently

$$\begin{aligned} |y_{n+1} - ab| &= |y_{n+1} - x_0 z_0| \\ &= |(y_0 + (z_0 - z_{n+1})x_0) - x_0 z_0| \\ &= |0 - z_{n+1} x_0| \\ &\leq 2^{-n} \cdot 2 = 2^{-n+1} \end{aligned}$$

## An Example: Computing sin and cos

To compute $\sin\theta$ and $\cos\theta$ for $|\theta| \leq \pi/2$ we let

$$m = 1,\ \varepsilon_k = \arctan(2^{-k}).$$

We also define

$$C = \prod_{k=0}^{n} \cos(\varepsilon_k).$$

The scheme becomes

$$\begin{aligned}
x_{k+1} &= x_k - \delta_k y_k 2^{-k} && (13)\\
y_{k+1} &= y_k + \delta_k x_k 2^{-k} && (14)\\
z_{k+1} &= z_k - \delta_k \varepsilon_k && (15)\\
\delta_k &= \operatorname{sgn}(z_k) && (16)\\
x_0 &= C && (17)\\
y_0 &= 0 && (18)\\
z_0 &= \theta && (19)
\end{aligned}$$

## Theorem

### The Cordic Representation Theorem

*Suppose* $(\varepsilon_k)_{k=0}^{n}$ *is a decreasing sequence of positive real numbers satisfying*

$$\varepsilon_k \leq \left( \sum_{j=k+1}^{n} \varepsilon_j \right) + \varepsilon_n$$

*for* $k = 0, 1, \ldots, n$, *and suppose* $r$ *is a real number such that*

$$|r| \leq \sum_{k=0}^{n} \varepsilon_j.$$

*If* $s_0 = 0$, *and* $s_{k+1} = s_k + \delta_k \varepsilon_k$, *for* $k = 0, 1, \ldots, n$, *where* $\delta_k = \operatorname{sgn}(r - s_k)$. *Then*

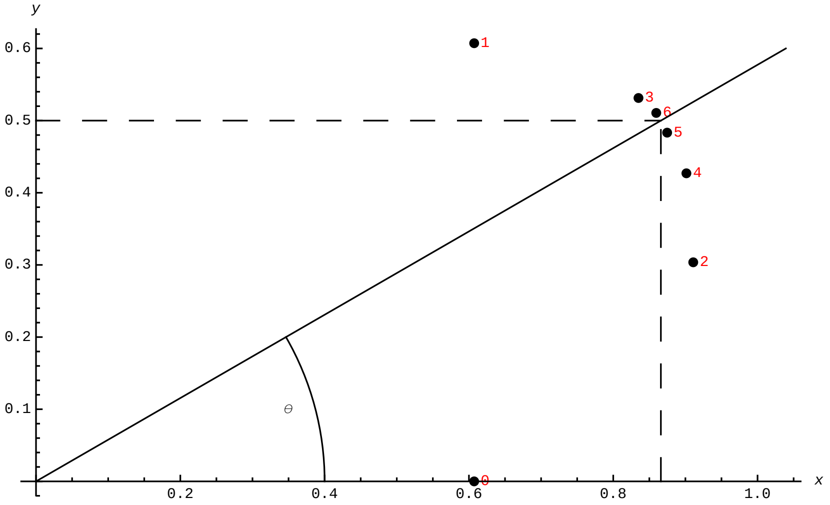
$$|r - s_{n+1}| = |r - \sum_{k=0}^{n} \delta_k \varepsilon_k| < \varepsilon_n.$$

**Proof:** By induction.

The previous theorem and some computations tell us that it is possible to write the angle $r = \theta$ as a combination of the angles $\varepsilon_k$, more precisely, we can choose $\delta_k = \pm 1$ so that

$$|\theta - \sum_{k=0}^{n} \delta_k \varepsilon_k| < \varepsilon_n.$$

Set $s_{n+1} = \sum_{k=0}^{n} \delta_k \varepsilon_k$.


y
0.6
0.5
0.4
0.3
0.2
0.1
0.2
0.4
0.6
0.8
1.0
x
θ
0
1
2
3
4
5
6

We then have the following

**Theorem**

*If* $x_0 = \prod_{k=0}^{n} \cos \varepsilon_k$ *and* $y_0 = 0$, *then* $x_{n+1} = \cos s_{n+1}$ *and* $y_{n+1} = \sin s_{n+1}$.

**Proof (by induction, sketch):**

$$\begin{aligned}
x_{n+1} &= x_n - \delta_n y_n 2^{-n} \\
&= (\cos s_n \cos \varepsilon_n) - \delta_n (\sin s_n \cos \varepsilon_n) \tan \varepsilon_n \\
&= \cos s_n \cos \varepsilon_n - \delta_n \sin s_n \sin \varepsilon_n \\
&= \cos(s_n + \delta_n \varepsilon_n) \\
&= \cos(s_{n+1})
\end{aligned}$$

## Trigonometric Functions

$\underline{m = 1}$

1. $x_{n+1} \approx \cos\theta,\ y_{n+1} \approx \sin\theta$:

   $\varepsilon_k = \arctan(2^{-k}),\ \delta_k = \operatorname{sgn} z_k,$
   $x_0 = \prod_{k=0}^{n} \cos(\varepsilon_k),\ y_0 = 0,\ z_0 = \theta$

2. $z_{n+1} \approx \arctan(y_0/x_0)$:

   $\varepsilon_k = \arctan(2^{-k}),\ \delta_k = -\operatorname{sgn} y_k,$
   $z_0 = 0$

## Hyperbolic Functions

$\underline{m = -1}$

1. $x_{n+1} \approx \cosh\theta$, $y_{n+1} \approx \sinh\theta$,
   and thus $x_{n+1} + y_{n+1} \approx e^{\theta}$:

   $\varepsilon_k = \operatorname{arctanh}(2^{-k})$ (repeated), $\delta_k = \operatorname{sgn} z_k$,
   $x_0 = C'$, $y_0 = 0$, $z_0 = \theta$

2. $z_{n+1} \approx \operatorname{arctanh}(y_0/x_0)$, $x_{n+1} \approx C'\sqrt{x_0^2 - y_0^2}$,

   and thus we obtain an approximation of $\ln x = 2 \ \operatorname{arctanh}\left(\frac{x-1}{x+1}\right)$:

   $\varepsilon_k = \operatorname{arctanh}(2^{-k})$ (repeated), $\delta_k = -\operatorname{sgn} y_k$,
   $z_0 = 0$

## Multiplication and Division

$\underline{m = 0}$

1. $y_{n+1} \approx x_0 z_0$:

   $\varepsilon_k = 2^{-k}$, $\delta_k = \operatorname{sgn} z_k$,
   $y_0 = 0$

2. $z_{n+1} \approx y_0 / x_0$:

   $\varepsilon_k = 2^{-k}$, $\delta_k = -\operatorname{sgn} y_k$,
   $z_0 = 0$

## References

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