How Do Calculators Calculate?

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April 25, 1997

We give an introduction to the

CORDIC method

used my most handheld calculators (such as the ones by Texas Instruments and Hewlett-Packard) to approximate the standard transcendental functions.

The CORDIC algorithm does not use Calculus based methods such as polynomial or rational function approximation.

The CORDIC (= **CO**ordinate **R**otation **DI**gital **C**omputer) algorithm was developed by

Jack E. Volder

in 1959.

His objective was to build a real-time navigational computer for use on aircrafts, so he was primarily interested in computing trigonometric functions.

Subsequently, the CORDIC scheme was extended by

J. Walther

in 1971 to other transcendental functions.

Hand-held calculators do not convert numbers to base 2.

They use a binary-coded decimal (BCD) system instead.

Calculators can only perform four operations inexpensively:

1. **Addition** and **Subtraction**

2. **Storing** in memory and **Retrieving** from memory

3. **Digit shift** (multiplication/division by the base)

4. **Comparisons**

The CORDIC Algorithm is a unified computational scheme to perform

1. multiplication and division

2. computations of the trigonometric functions

$$\sin, \cos, \arctan$$

3. computations of the hyperbolic trigonometric functions

$$\sinh, \cosh, \operatorname{arctanh}$$

4. and consequently can also compute the exponential function, the natural logarithm and the square root

The CORDIC Algorithm (Binary Version)

$$\begin{aligned} x_{k+1} &= x_k - m\delta_k y_k 2^{-k} && (1) \\ y_{k+1} &= y_k + \delta_k x_k 2^{-k} && (2) \\ z_{k+1} &= z_k - \delta_k \varepsilon_k && (3) \end{aligned}$$

$$\delta_k = \pm 1 \text{ (depending on } y_k \text{ or } z_k) \qquad (4)$$

$$m = 0, \pm 1 \qquad (5)$$

$$\varepsilon_k \text{ are prestored constants} \qquad (6)$$

A Warm-Up Example: Multiplication

To compute the product of a and b for $|a| \leq 2$ and $|b| \leq 2$ we let

$$m = 0,\ \varepsilon_k = 2^{-k}.$$

The scheme becomes

$$\begin{aligned} x_{k+1} &= x_k \quad (= x_0) && (7) \\ y_{k+1} &= y_k + \delta_k x_k 2^{-k} && (8) \\ z_{k+1} &= z_k - \delta_k 2^{-k} && (9) \end{aligned}$$

$$\delta_k = \mathrm{sgn}(z_k) \qquad (10)$$

$$\begin{aligned} x_0 &= a && (11) \\ y_0 &= 0 && (12) \\ z_0 &= b && (13) \end{aligned}$$

From the previous examples we suspect that $z_n \to 0$ and that $y_n \to ab$. Let's check:

Claim: $|z_{n+1}| \leq 2^{-n}$

Proof by induction: By the restriction placed on b

$$|z_0| = |b| \leq 2^{-(-1)} = 2.$$

Now suppose $|z_n| \leq 2^{-(n-1)} = 2^{-n+1}$. By definition $z_{n+1} = z_n - \mathrm{sgn}(z_n)2^{-n}$. There are two cases to consider:

If $0 \leq z_n \leq 2^{-n+1}$, then $\mathrm{sgn}(z_n) = 1$, and consequently $-2^{-n} \leq z_{n+1} \leq 2^{-n}$.

If on the other hand $-2^{-n+1} \leq z_n < 0$, then $\mathrm{sgn}(z_n) = -1$, and consequently $-2^{-n} \leq z_{n+1} \leq 2^{-n}$.

Claim: $|y_{n+1} - ab| \leq 2^{-n+1}$

Proof: Note that the iterative formulas imply:

$$z_{n+1} = z_0 - \sum_{k=0}^{n} \delta_k 2^{-k},$$

and

$$y_{n+1} = y_0 + \left(\sum_{k=0}^{n} \delta_k 2^{-k} \right) x_0.$$

Consequently

$$\begin{aligned}
|y_{n+1} - ab| &= |y_{n+1} - x_0 z_0| \\
&= \left| \Big(y_0 + (z_0 - z_{n+1}) x_0 \Big) - x_0 z_0 \right| \\
&= |0 - z_{n+1} x_0| \\
&\leq 2^{-n} \cdot 2 = 2^{-n+1}
\end{aligned}$$

An Example: Computing sin and cos

To compute $\sin\theta$ and $\cos\theta$ for $|\theta| \leq \pi/2$ we let

$$m = 1,\ \varepsilon_k = \arctan(2^{-k}).$$

We also define

$$C = \prod_{k=0}^{n} \cos(\varepsilon_k).$$

The scheme becomes

$$\begin{aligned}
x_{k+1} &= x_k - \delta_k y_k 2^{-k} && (14)\\
y_{k+1} &= y_k + \delta_k x_k 2^{-k} && (15)\\
z_{k+1} &= z_k - \delta_k \varepsilon_k && (16)\\
\delta_k &= \mathrm{sgn}(z_k) && (17)\\
x_0 &= C && (18)\\
y_0 &= 0 && (19)\\
z_0 &= \theta && (20)
\end{aligned}$$

The Cordic Representation Theorem

Suppose $(\varepsilon_k)_{k=0}^n$ *is a decreasing sequence of positive real numbers satisfying*

$$\varepsilon_k \le \left(\sum_{j=k+1}^{n} \varepsilon_j\right) + \varepsilon_n$$

for $k = 0, 1, \ldots, n$, *and suppose* r *is a real number such that*

$$|r| \le \sum_{k=0}^{n} \varepsilon_j.$$

If $s_0 = 0$, *and* $s_{k+1} = s_k + \delta_k \varepsilon_k$, *for* $k = 0, 1, \ldots, n$, *where*

$$\delta_k = \mathrm{sgn}\left(\mathrm{r} - \mathrm{s_k}\right),$$

then

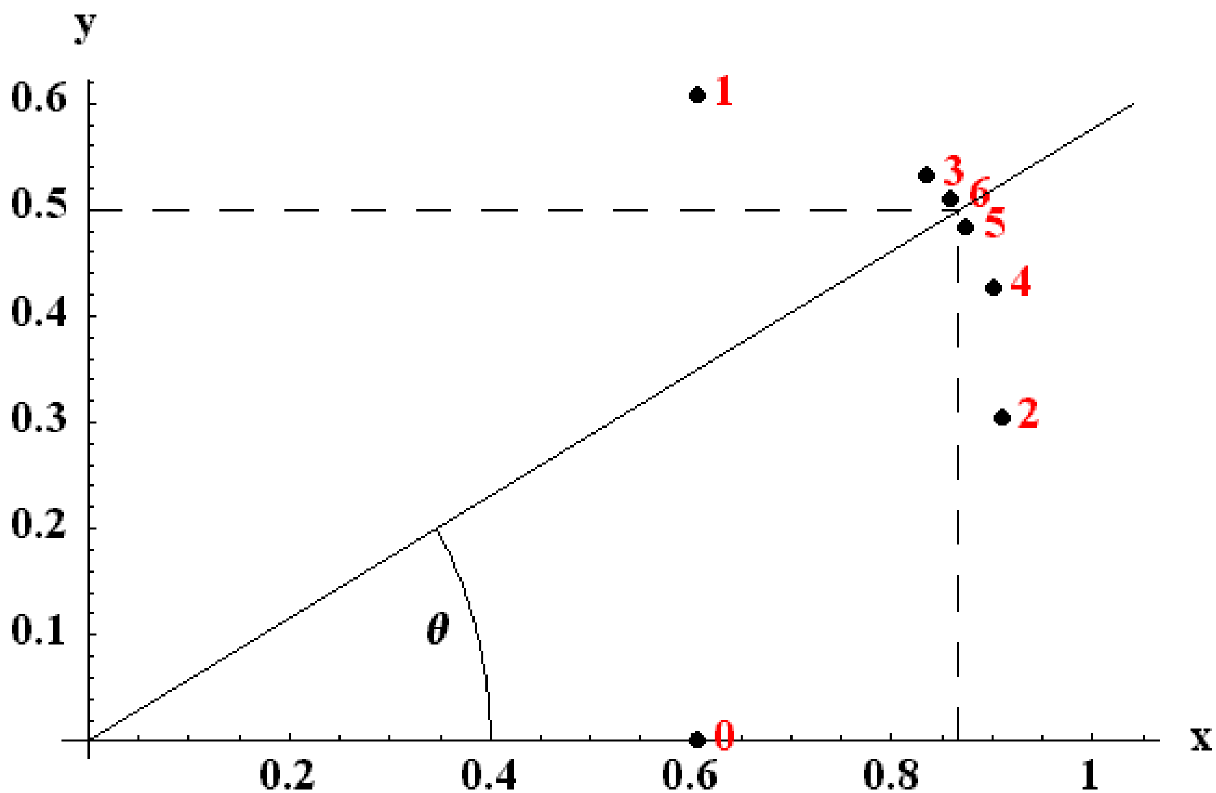
$$|r - s_{n+1}| = |r - \sum_{k=0}^{n} \delta_k \varepsilon_k| < \varepsilon_n.$$

Proof: By induction.

The previous Theorem and some computations tell us that it is basically possible to write the angle $r = \theta$ as a combination of the angles ε_k, more precisely, we can choose $\delta_k = \pm 1$ so that

$$|\theta - \sum_{k=0}^{n} \delta_k \varepsilon_k| < \varepsilon_n.$$

Set $s_{n+1} = \sum_{k=0}^{n} \delta_k \varepsilon_k$.


y
0.6
0.5
0.4
0.3
0.2
0.1
1
3
6
5
4
2
θ
0
x
0.2
0.4
0.6
0.8
1

We then have the following

Theorem. If $x_0 = \prod_{k=0}^{n} \cos \varepsilon_k$ and $y_0 = 0$, then $x_{n+1} = \cos s_{n+1}$ and $y_{n+1} = \sin s_{n+1}$.

Proof (by induction, sketch):

$$\begin{aligned}
x_{n+1} &= x_n - \delta_n y_n 2^{-n} \\
&= (\cos s_n \cos \varepsilon_n) - \delta_n (\sin s_n \cos \varepsilon_n) \tan \varepsilon_n \\
&= \cos s_n \cos \varepsilon_n - \delta_n \sin s_n \sin \varepsilon_n \\
&= \cos(s_n + \delta_n \varepsilon_n) \\
&= \cos(s_{n+1})
\end{aligned}$$

Trigonometric Functions

$\underline{m = 1}$

1. $x_{n+1} \approx \cos\theta$, $y_{n+1} \approx \sin\theta$:

   $\varepsilon_k = \text{arctan}(2^{-k})$, $\delta_k = \text{sgn} z_k$,

   $x_0 = \prod_{k=0}^{n} \cos(\varepsilon_k)$, $y_0 = 0$, $z_0 = \theta$

2. $z_{n+1} \approx \text{arctan}(y_0/x_0)$:

   $\varepsilon_k = \text{arctan}(2^{-k})$, $\delta_k = -\text{sgn} y_k$,

   $z_0 = 0$

Hyperbolic Functions

$\underline{m = -1}$

1. $x_{n+1} \approx \cosh\theta$, $y_{n+1} \approx \sinh\theta$,
   and thus $x_{n+1} + y_{n+1} \approx e^{\theta}$:
   $\varepsilon_k = \operatorname{arctanh}(2^{-k})$ (repeated), $\delta_k = \operatorname{sgn} z_k$,
   $x_0 = C'$, $y_0 = 0$, $z_0 = \theta$

2. $z_{n+1} \approx \operatorname{arctanh}(y_0/x_0)$, $x_{n+1} \approx C'\sqrt{x_0^2 - y_0^2}$,
   and thus we obtain an approximation of $\ln x = 2\ \operatorname{arctanh}\left(\dfrac{x-1}{x+1}\right)$:
   $\varepsilon_k = \operatorname{arctanh}(2^{-k})$ (repeated), $\delta_k = -\operatorname{sgn} y_k$,
   $z_0 = 0$

Multiplication and Division

$\underline{m = 0}$

1. $y_{n+1} \approx x_0 z_0$:
   $\varepsilon_k = 2^{-k}$, $\delta_k = \mathrm{sgn} z_k$,
   $y_0 = 0$

2. $z_{n+1} \approx y_0 / x_0$:
   $\varepsilon_k = 2^{-k}$, $\delta_k = -\mathrm{sgn} y_k$,
   $z_0 = 0$

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