# Exploring Machin's Approximation of $\pi$

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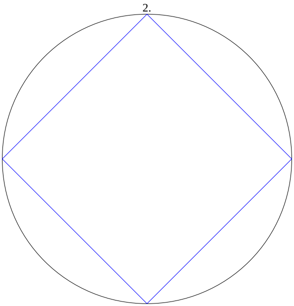
**Archimedes of Syracuse** ($\approx$ 287–212 BC) approximated $\pi$ by the Method of Exhaustion:



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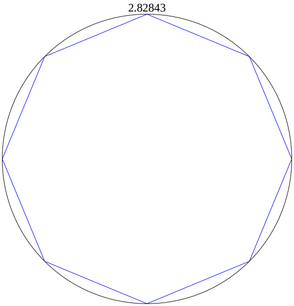




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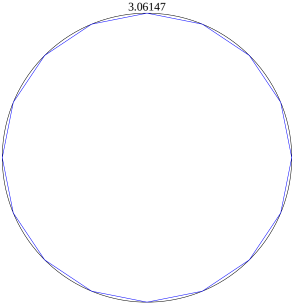

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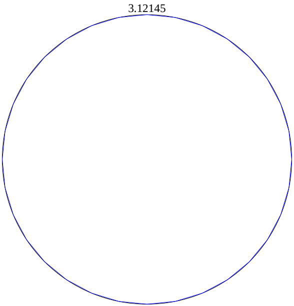
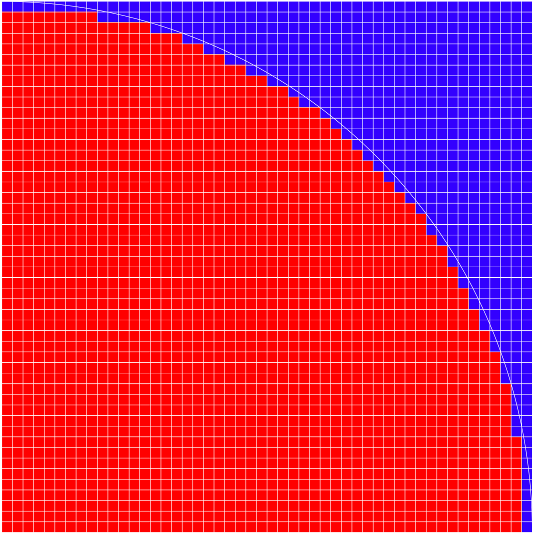



The accuracy of the approximation increases by about 1 digit for every 2 iterations.

3.0576





**James Gregory** (1638–1675) found the McLaurin series expansion for the arctangent function:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \ldots$$

The radius of convergence of the series is 1.



As a consequence we obtain **Leibniz' Formula**:

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

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Convergence of this series is very slow:

Summing 10,000 terms of the series yields 3.14149 as an approximation of $\pi$.



This is a nice illustration of the **General Calculus Philosophy**:

> *Convergence behavior is good, if $x$ is close to the center of a Taylor series; convergence is bad if $x$ is close to the "edge" of convergence.*



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Therefore **John Machin** ($\approx$ 1686-1751) [and others] had the following idea:

> *Replace the 1 in Leibniz' Formula by one or more numbers closer to 0.*



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The ingredient needed is the **addition formula** for the tangent function:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$



Applying the addition formula twice we obtain:

$$\tan(4\alpha) = \frac{4\tan(\alpha) - 4\tan^3(\alpha)}{1 + \tan^4(\alpha) - 6\tan^2(\alpha)}$$



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If we set $x = \tan\alpha$ and $y = \tan\beta$, we can apply the addition formula one more time to get

$$\tan(4\alpha + \beta) = \frac{x^4 y - 4x^3 - 6x^2 y + 4x + y}{x^4 + 4x^3 y - 6x^2 - 4xy + 1}$$



Using this for

$$\frac{\pi}{4} = 4\alpha + \beta,$$

we get the equation:

$$1 = \frac{x^4 y - 4x^3 - 6x^2 y + 4x + y}{x^4 + 4x^3 y - 6x^2 - 4xy + 1}$$



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At this point, Machin made the decision to choose $x = \frac{1}{5}$, then solved for $y$ to obtain $y = -\frac{1}{239}$.



Since the angles $\alpha = \arctan\left(\frac{1}{5}\right)$ and $\beta = \arctan\left(-\frac{1}{239}\right)$ satisfy the equation

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Machin then used this formula in 1706 to compute 100 digits of $\pi$.



Using similar ideas and identities such as

$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12943},$$

found by **Carl Størmer** in 1896,

**Yasumasa Kanada** (1949–) and his collaborators computed more than 1.24 trillion digits of $\pi$ in 2002.



References:


- Jack S. Calcut: *Gaussian Integers and Arctangent Identities for $\pi$.* American Mathematical Monthly 116 (2009), 515–530.
- Jonathan M. Borwein: *The Life of Pi.* `http://www.cecm.sfu.ca/~jborwein/pi-slides.pdf`
- David Blatner: *The Joy of Pi.* `http://www.joyofpi.com/pilinks.html`