Discrete Wavelets and Image Processing

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Course Objectives:

- Get a flavor of the ideas and issues involved in applying mathematics to a relevant engineering problem
- Develop an understanding of the theoretical underpinnings of wavelet transforms and their applications
- Learn how to use a computer algebra system for mathematical investigations, as a computational and visualization aid, and for the implementation of mathematical algorithms
Prerequisites:

- A thorough understanding of Calculus
- Some familiarity with matrices
- Mathematical maturity
- Willingness to learn Mathematica
The Engineering Problem:
- Transmit digital information through a “narrow channel”.
- “Lossy compression”: The information received need not be identical to the one sent, but the quality must be “acceptable”.
Applications:

- Photos
- MP3 players
- Real-time two-way audio (cellular telephones)
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- Streaming video (Netflix, Hulu)
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- **Photos**
- MP3 players
- Real-time two-way audio (cellular telephones)
- Streaming video (Netflix, Hulu)
- Real-time two-way audio and video (Skype)
A color image consists of three color channels: Red, Green and Blue
Each pixel in a gray-scale image is represented by an integer between 0 and $2^8 - 1$ (8 bit = 1 byte).

$0=$black, $255=$white
“Raw” storage requirement:
\[(512 \times 768) \text{ bytes} = 393.2 \text{ KB}\]
“Naive compression” — Average of four neighboring pixels: Compression factor: 4
Mathematical Question: What ways are there to approximate a function ("data") other than just averaging?
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- Differentiation Techniques
  - Taylor series
- Integration Techniques
  - Fourier series
  - Wavelets
0.827958 \sin(t) - 0.310564 \sin(2t) + 0.191515 \sin(3t) - 0.139372 \sin(4t) + 0.109891 \sin(5t) - 0.0908419 \sin(6t) + 0.586021 \cos(t) - 0.172359 \cos(2t) + 0.079192 \cos(3t) - 0.0450785 \cos(4t) + 0.0290109 \cos(5t) - 0.0202076 \cos(6t) - 0.465052
Fourier series of the function $f(t)$:

$$\sum_{n=1}^{\infty} a_n \sin nt + \sum_{n=0}^{\infty} b_n \cos nt$$

The coefficients are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt \quad (n \geq 1)$$
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Approximating data

Fourier techniques

The first Fourier functions:
The JPEG algorithm uses Fourier techniques - it employs the “Discrete Cosine Fourier Transform (DCT)”.

Here is a JPEG example with compression factor 6:
Wavelet pioneers: Alfred Haar (t-r, on the left), Stephane Mallat (b-r), Ingrid Daubechies (t)
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The first Haar functions:
Wavelet approximation:

... using the Daubechies-4 wavelet
The basis functions are now re-scalings of the two functions below, the “father wavelet” (blue) and the “mother wavelet” (red).
Original image
1st color channel: $Y$
2nd color channel: $C_r$
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The JPEG2000 algorithm

3rd color channel: $C_b$
Applying the CDF97-wavelet transform to $Y$ once
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The JPEG2000 algorithm

...and again...
...and one more time:
Quantizing the $Y$ channel
The sender then encodes this “quantized image” and sends it through the narrow channel to the receiver.

The compression factor in this example is 10.2.

The receiver then decompresses the image to be able to view the approximation of the original image.
“Undoing” the transform (by receiver)
Original image, again...
Zooming in on the original image:
Zooming in on the received image:
Any Questions?