Lloyd-Max Quantization Schemes

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Learning Outcomes:

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- Students will reflect on the role of quantization in image compression.

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- Multi-variable Calculus
- Some knowledge of wavelet transforms and their use in image compression
- Some minimal statistics knowledge

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- Quantization is usually employed after transformation.

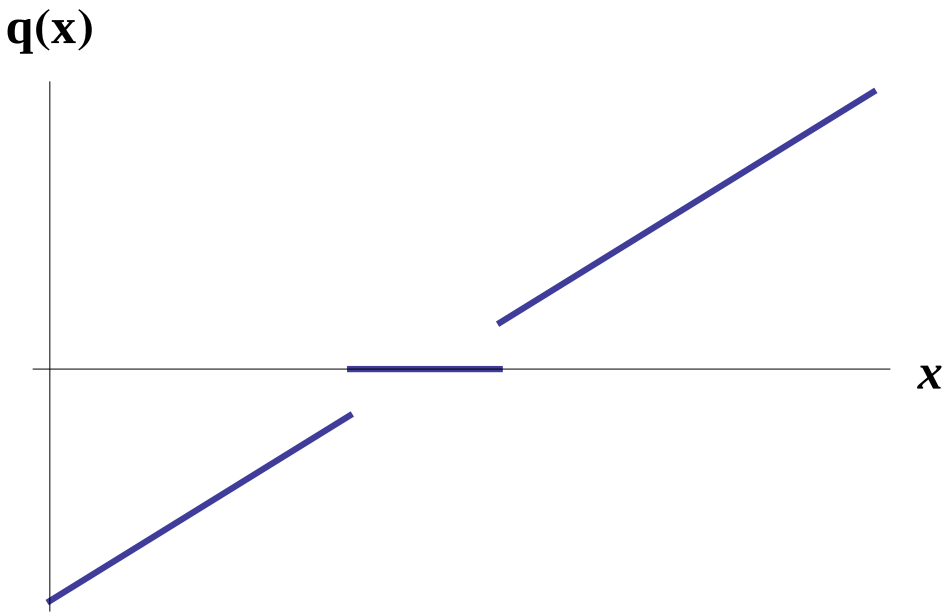
Basic Quantization Scheme: Thresholding

The most basic quantization technique is **Thresholding**:

Given a signal $\vec{x} = (x_i)$ and a single threshold σ, we replace values as follows:

$$q(x_i) = \begin{cases} 0 & \text{if } |x_i| \leq \sigma \\ x_i & \text{if } |x_i| > \sigma \end{cases}$$

The thresholding quantization function:



The **Lossy JPEG2000 Quantization Scheme** also has one fixed parameter, τ. After wavelet transformation, a "step" quantization

$$q(x_i) = sgn(x_i) \cdot \sigma \cdot \left\lfloor \frac{|x_i|}{\sigma} \right\rfloor$$

is applied to each region with a parameter σ determined as follows:

$\frac{\tau}{2}$	τ
τ	2τ

$n = 1$

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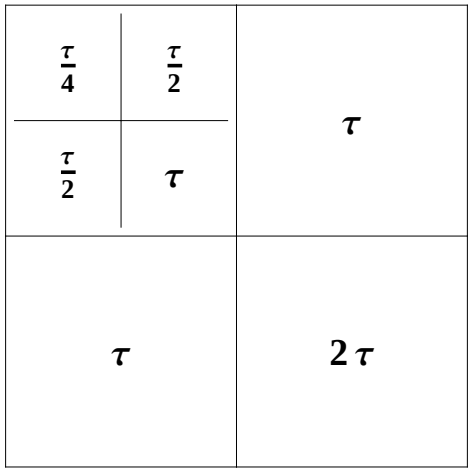

$n = 2$

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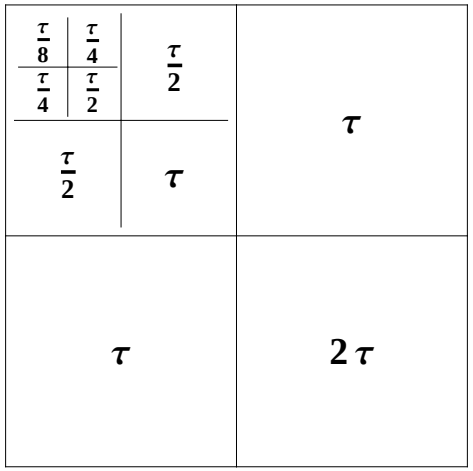
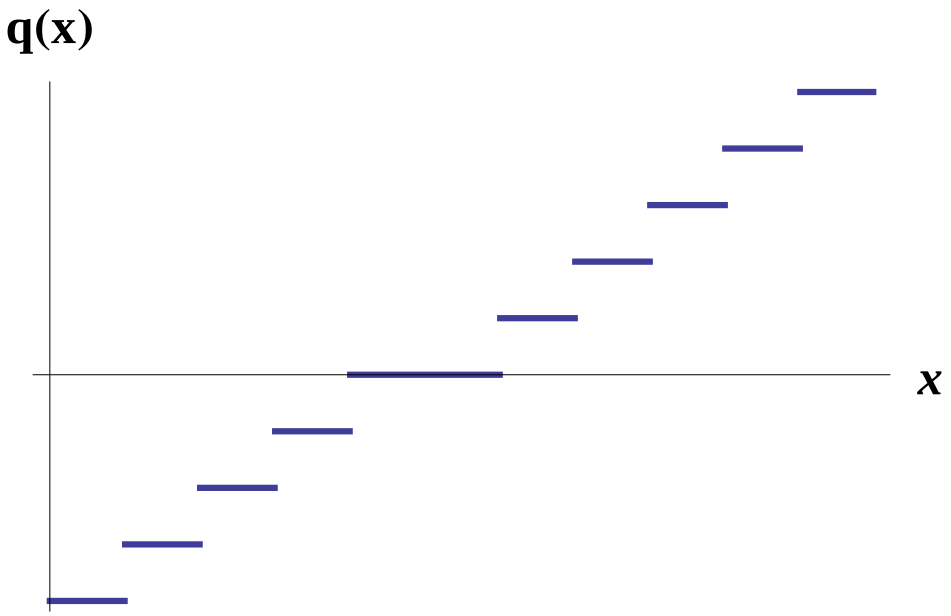


$n = 3$

The quantization function for a given region:



For a fixed signal $\vec{x}$ and a fixed positive integer n, **Lloyd-Max quantization** uses two sets of parameters:

- Bin-boundaries $\vec{L} = (L_1, L_2, \ldots, L_{n+1})$, with $\min \vec{x} = L_1 < L_2 < \cdots < L_n < L_{n+1} = 1 + \max \vec{x}$,
- and replacement values $\vec{p} = (p_1, p_2, \ldots, p_n)$.

The quantization function replaces the x-values in the bin $[L_j, L_{j+1})$ by the value p_j:

$$q(x_i) = p_j, \text{ when } x_i \in [L_j, L_{j+1})$$

The goal is to choose $\vec{L}$ and $\vec{p}$ to minimize the resulting quantization error

$$E(\vec{L}, \vec{p}) = \sum_{i=1}^{m} |x_i - q(x_i)|^2$$

This is a classical Calculus problem. Rewriting we obtain:

$$E(\vec{L}, \vec{p}) = \sum_{j=1}^{n} \sum_{x_i \in [L_j, L_{j+1})} (x_i - p_j)^2$$

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$$\frac{\partial E}{\partial p_j} = \sum_{x_i \in [L_j, L_{j+1})} 2(x_i - p_j) = 0 \Leftrightarrow p_j = \frac{\sum_{x_i \in [L_j, L_{j+1})} x_i}{\#\{i \mid x_i \in [L_j, L_{j+1})\}} \quad (1)$$

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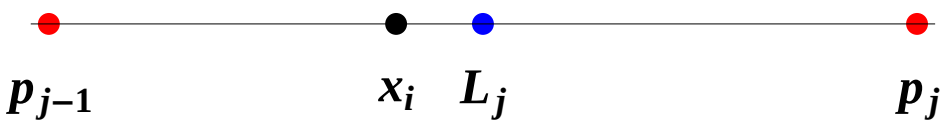

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- Equations (1) are used to update the values $\vec{p}$:

$$p_j^{\text{new}} = ave\ \left\{x_i \mid x_i \in [L_j, L_{j+1})\right\}$$

- Equations (2) then yield new values for $\vec{L}$:

$$L_j^{\text{new}} = \frac{1}{2}(p_{j-1}^{\text{new}} + p_j^{\text{new}})\ , \quad j = 2, \ldots, n$$

The values for L_1 and L_{n+1} are left unchanged.

As an example, we apply the algorithm to a grayscale image, with $n = 32$. The initial bins are chosen at random.

Original image, entropy=7.65

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21 iterations, entropy=4.75, PSNR 40.7

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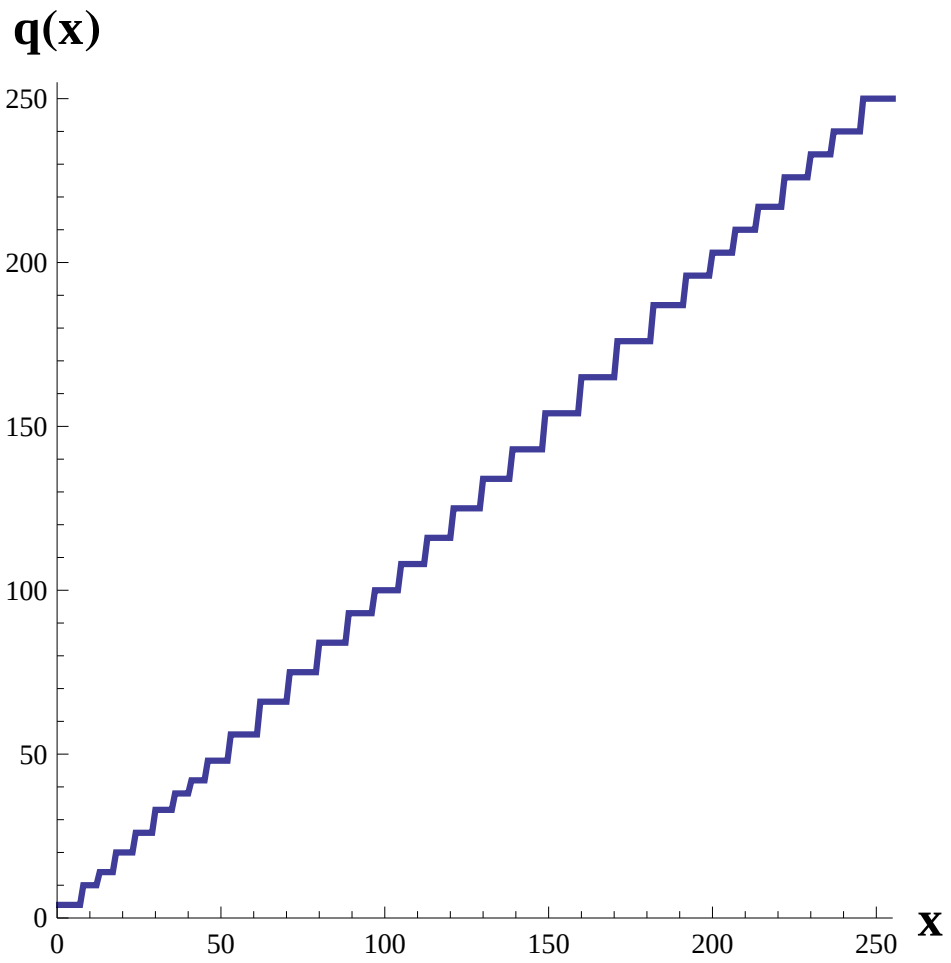
28 iterations, entropy=4.56, PSNR 40.0

Here is the quantization function for the second run:



Quantization applied to "raw" images usually gives bad results in areas of gradual gray-value change (sky, water, etc.):

Original image:
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$n = 2^6$:
10 iterations, entropy=5.53, PSNR 45.1

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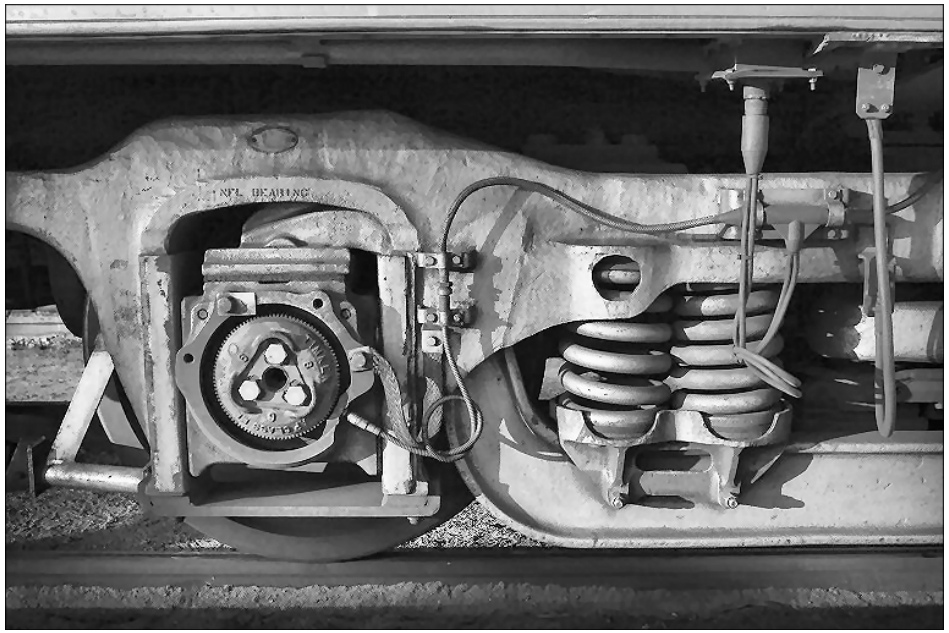
$n = 2^5$:
18 iterations, entropy=4.35, PSNR 38.0

Quantization applied to "raw" images usually gives bad results in areas of gradual gray-value change (sky, water, etc.):

$n = 2^4$:
22 iterations, entropy=3.76, PSNR 34.7

We now compare step quantization to Lloyd-Max quantization for a transformed image:

- We use the CDF97 wavelet transform once.
- For the step quantization we use $\tau = 16$:



Original image (-128): entropy=7.74

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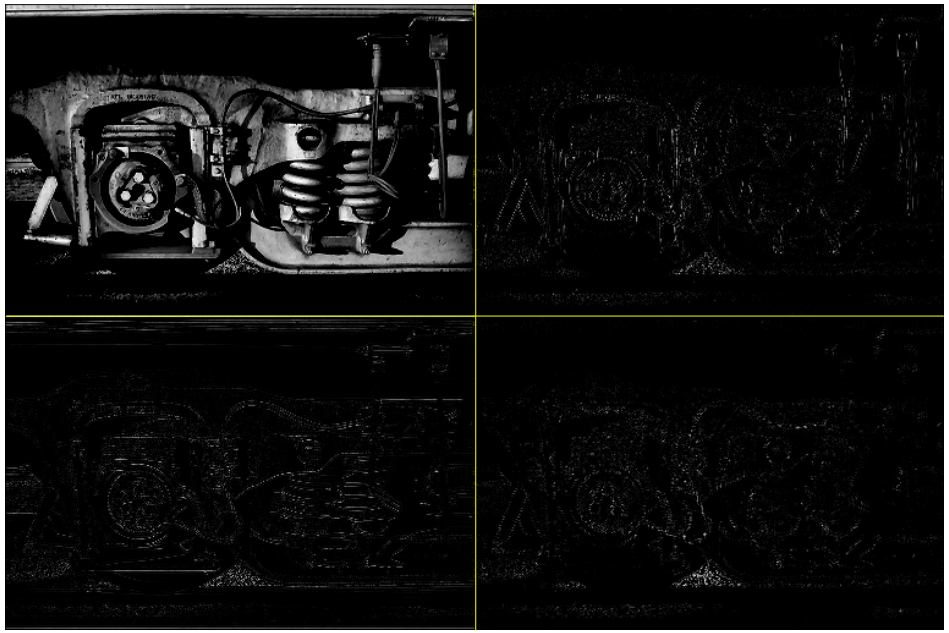

CDF97-transformed image

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$\tau/2=8$	$\tau=16$
$\tau=16$	$2\tau=32$

Step sizes for the quantization

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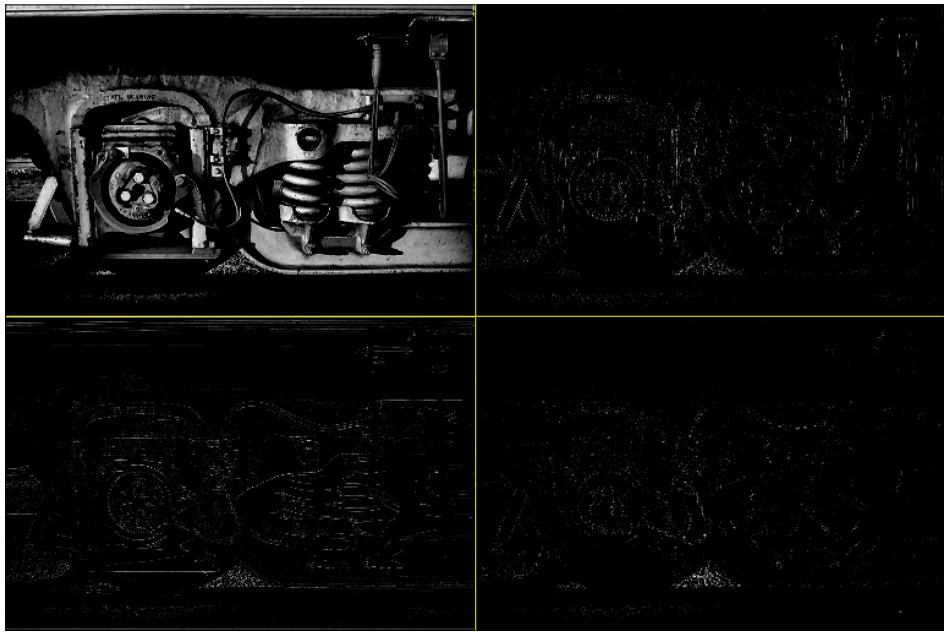

Quantized transform: entropy=2.75

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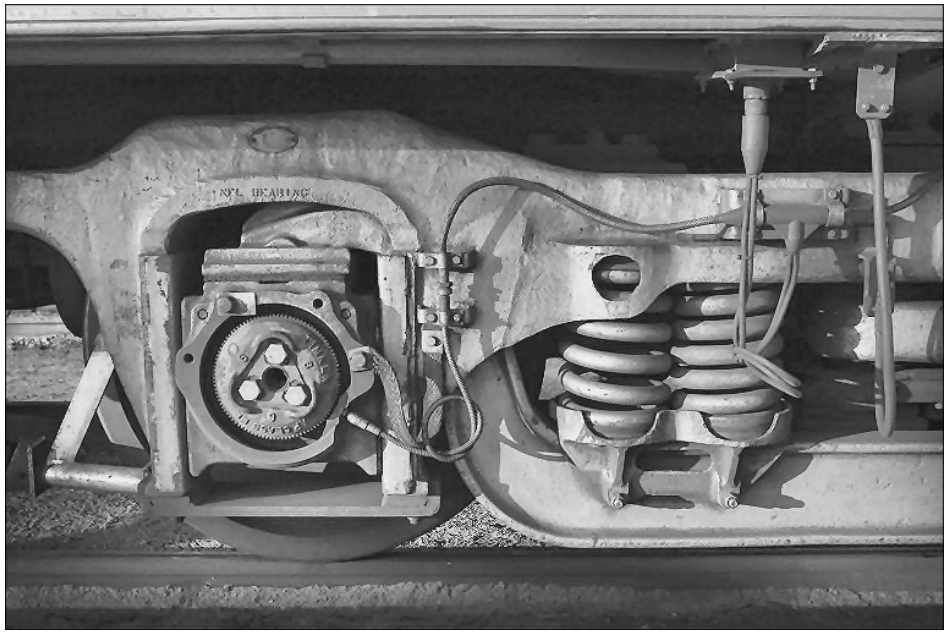
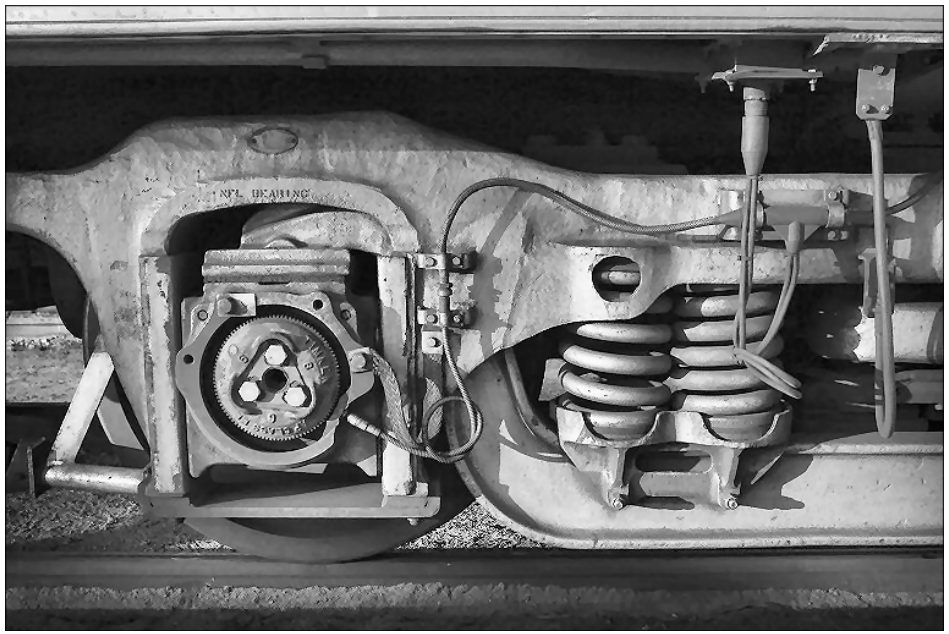


Reconstructed image: PSNR 32.44

Here is the same example with Lloyd-Max quantization:



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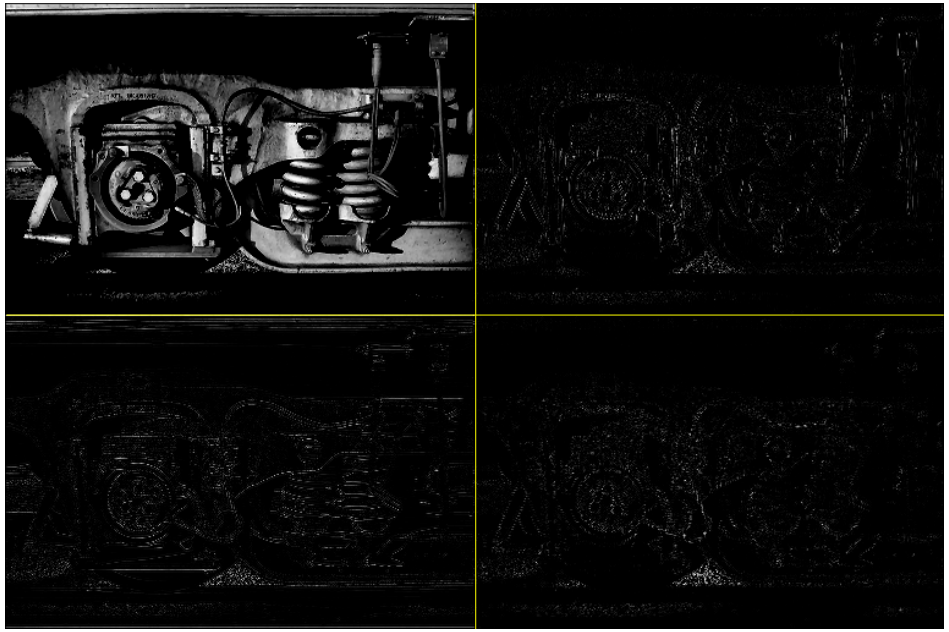

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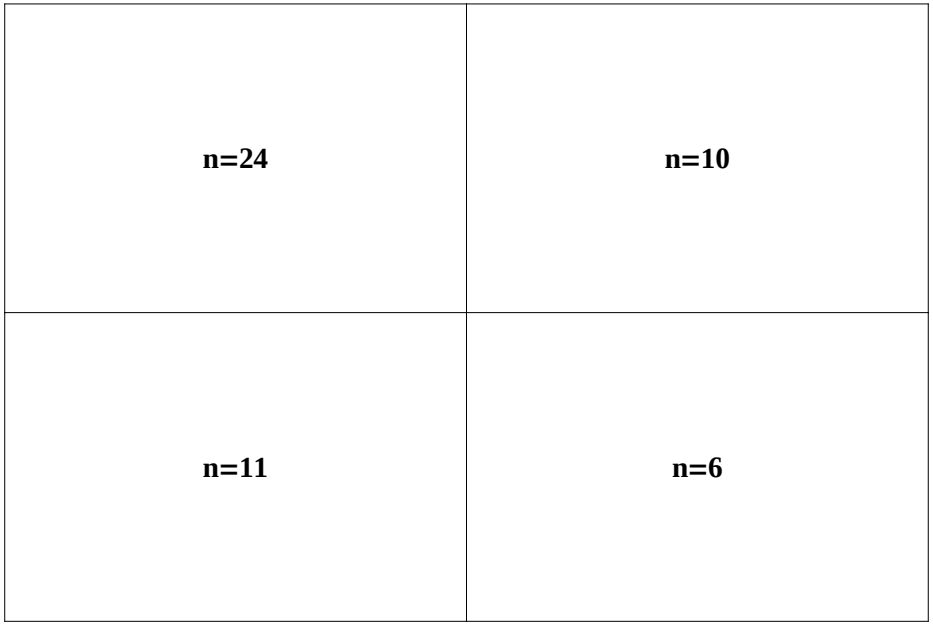

Number of bins for the quantization

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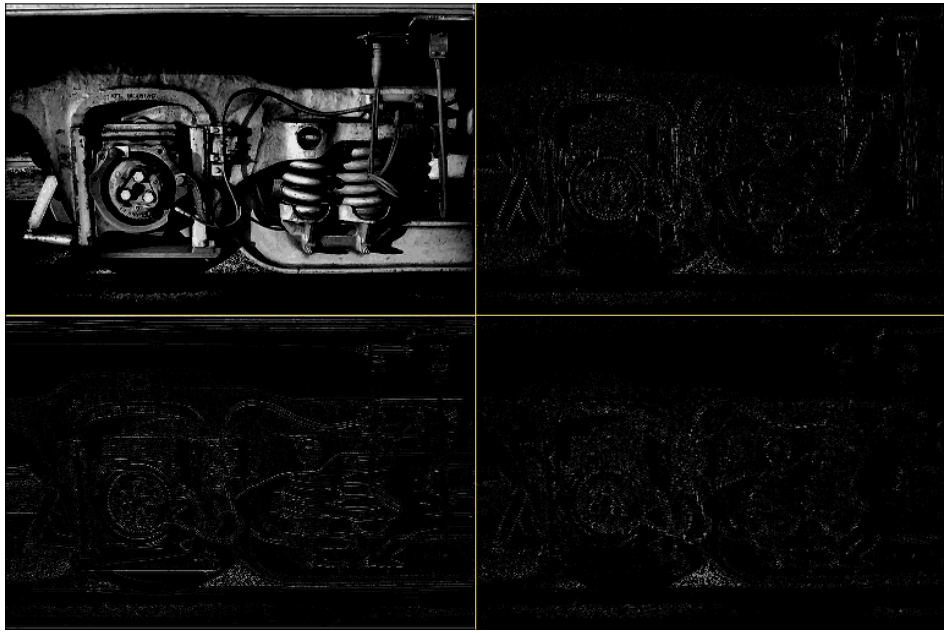

Quantized transform: entropy=3.73

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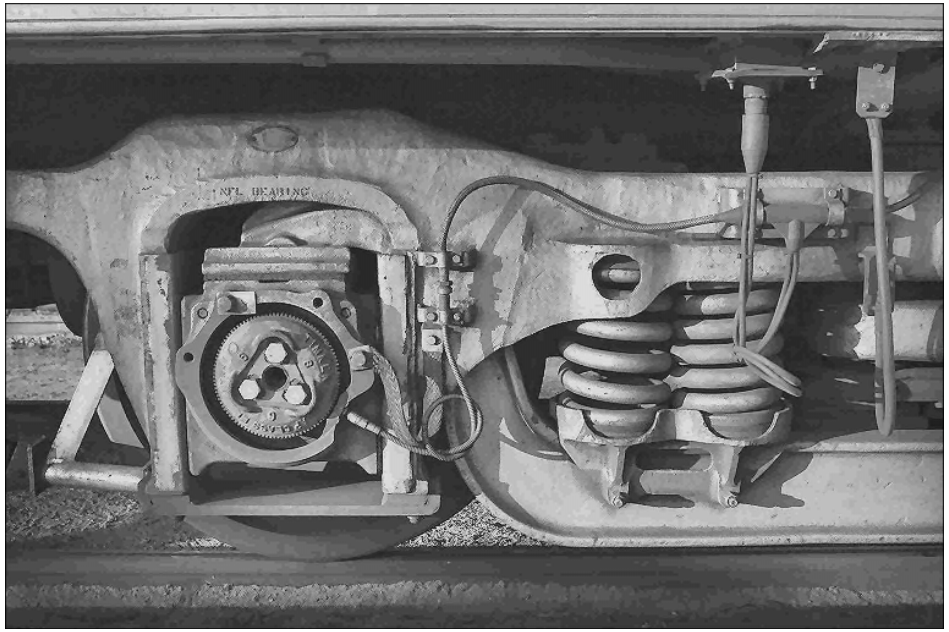
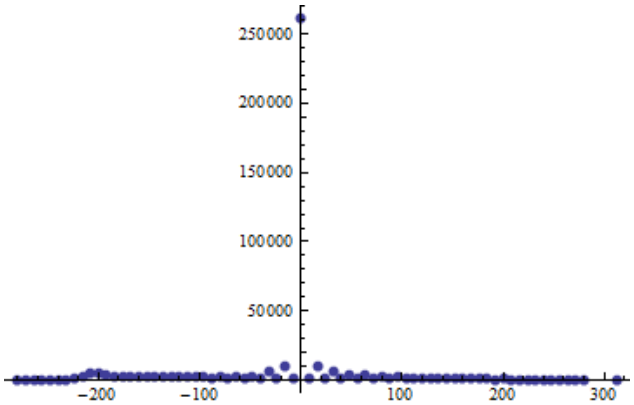
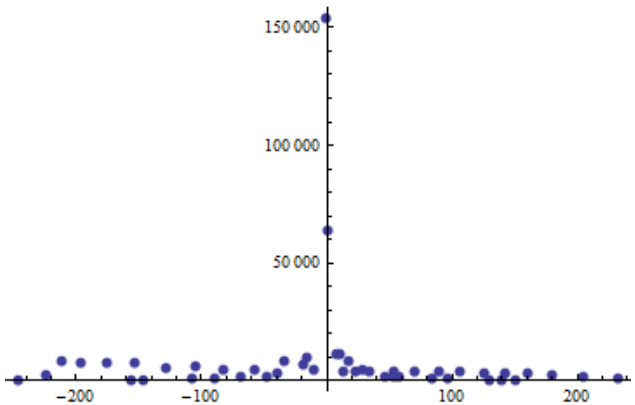


Reconstructed image: PSNR=33.44

Why is the entropy comparatively high after LM-quantization?
Here are the histograms of the quantized transformed signals:



Step quantization

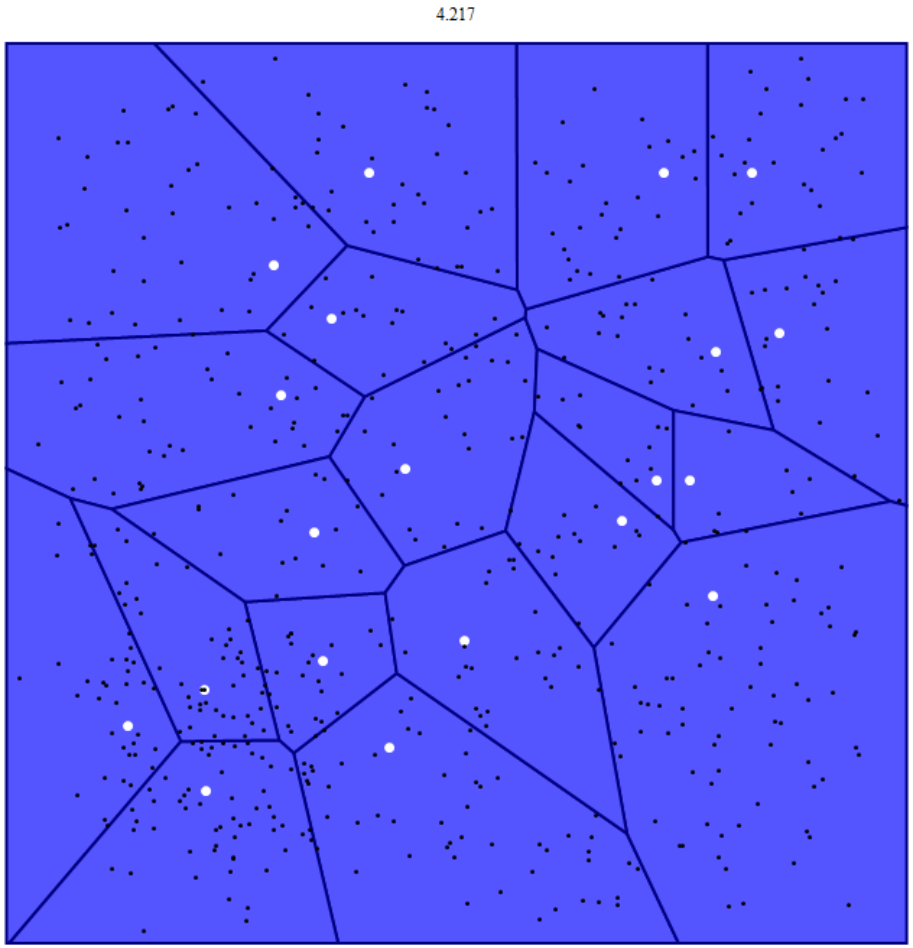


Lloyd-Max quantization

Lloyd-Max quantization can be generalized to higher dimensions.

- The bin boundary conditions in Equations (2) now become Voronoi cell conditions.
- The update step given by Equations (1) still remains: The new replacement values are the average of the signal values in the Voronoi cell.

Here is a two-dimensional example:



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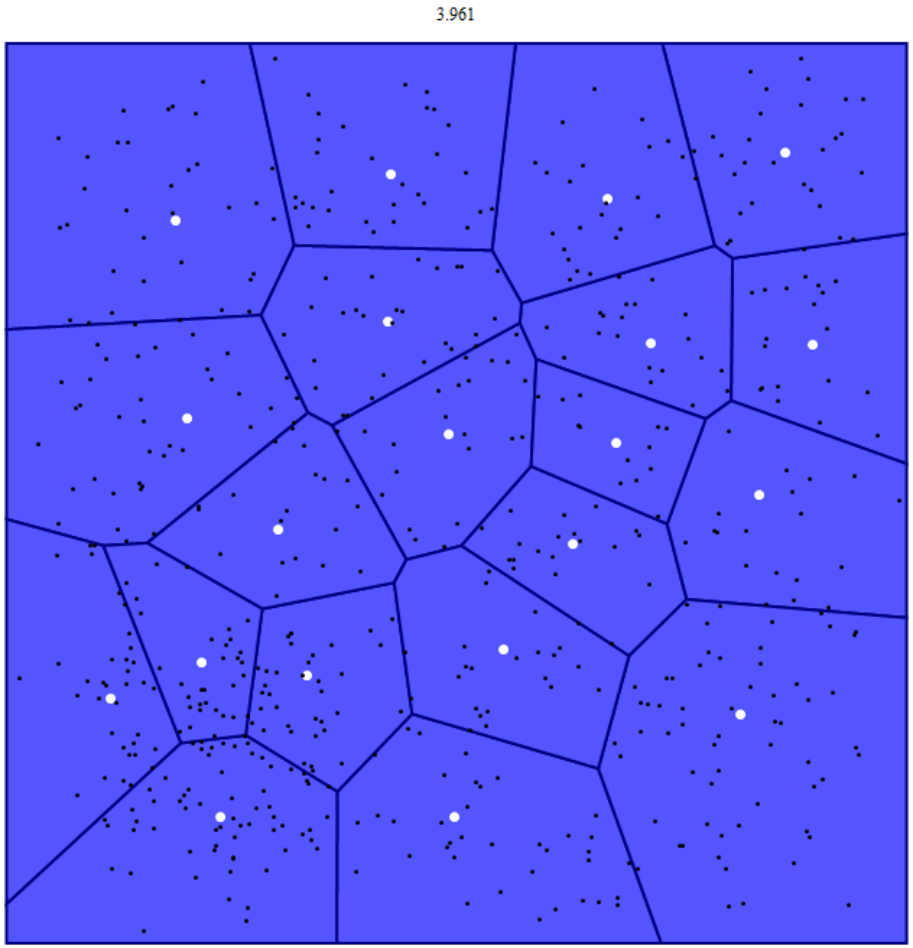

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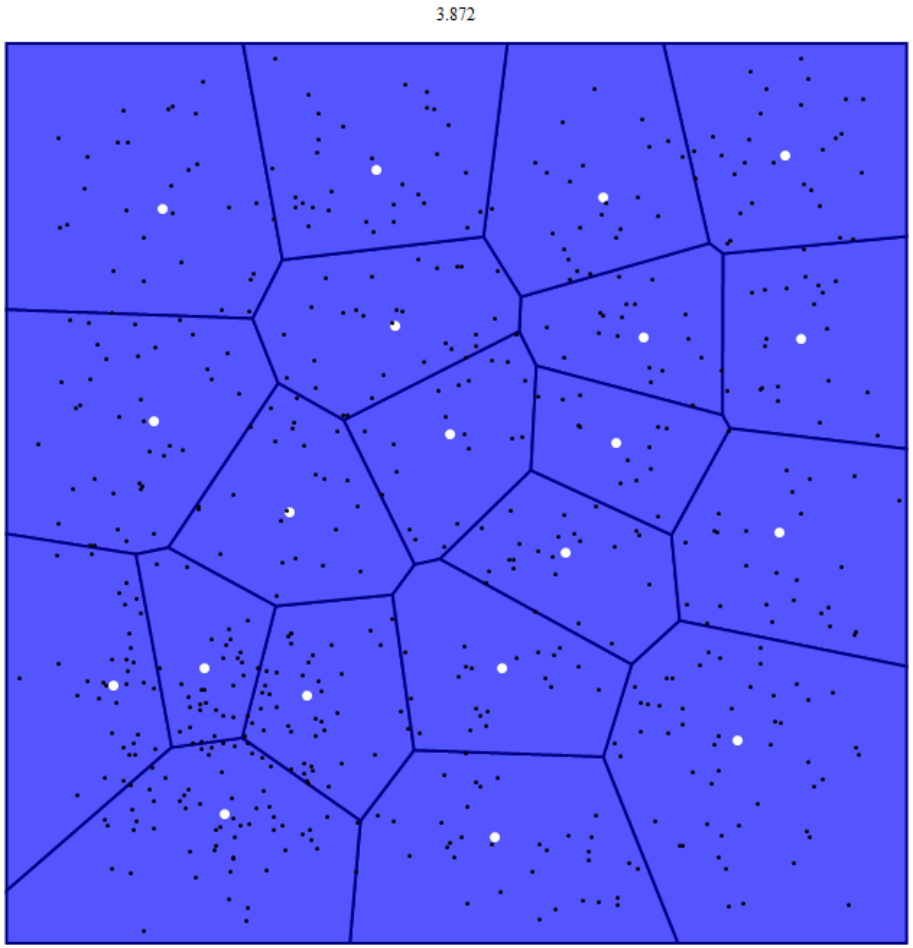
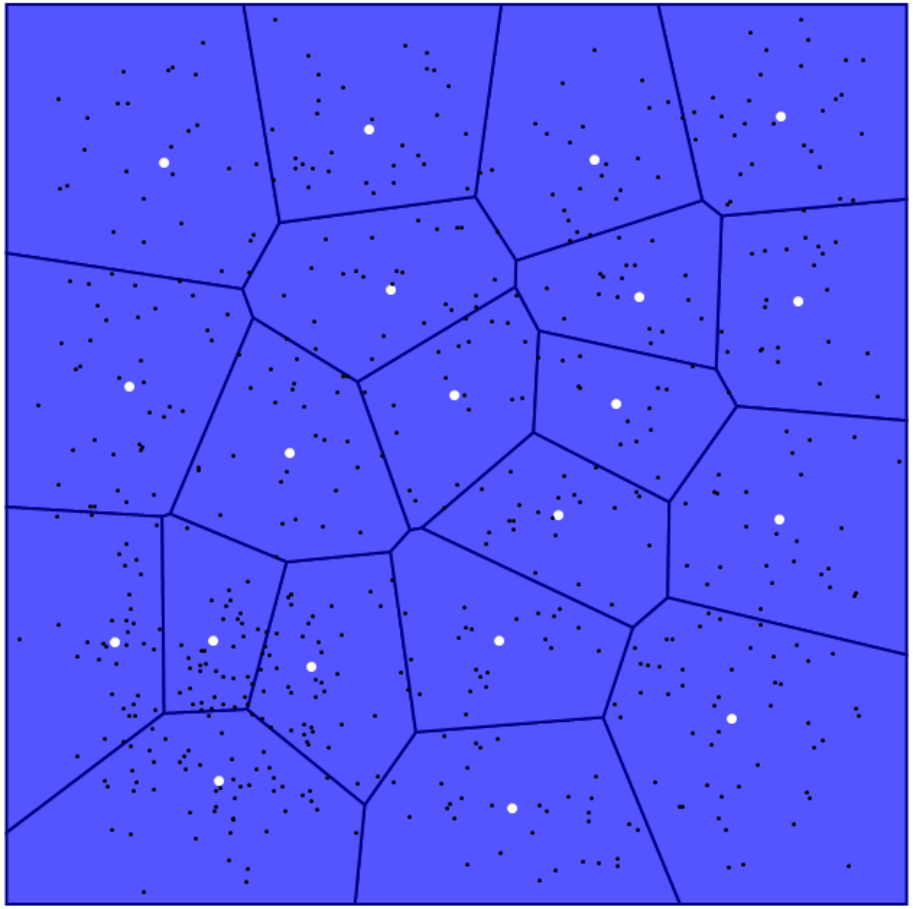


Here is a two-dimensional example:

3.831



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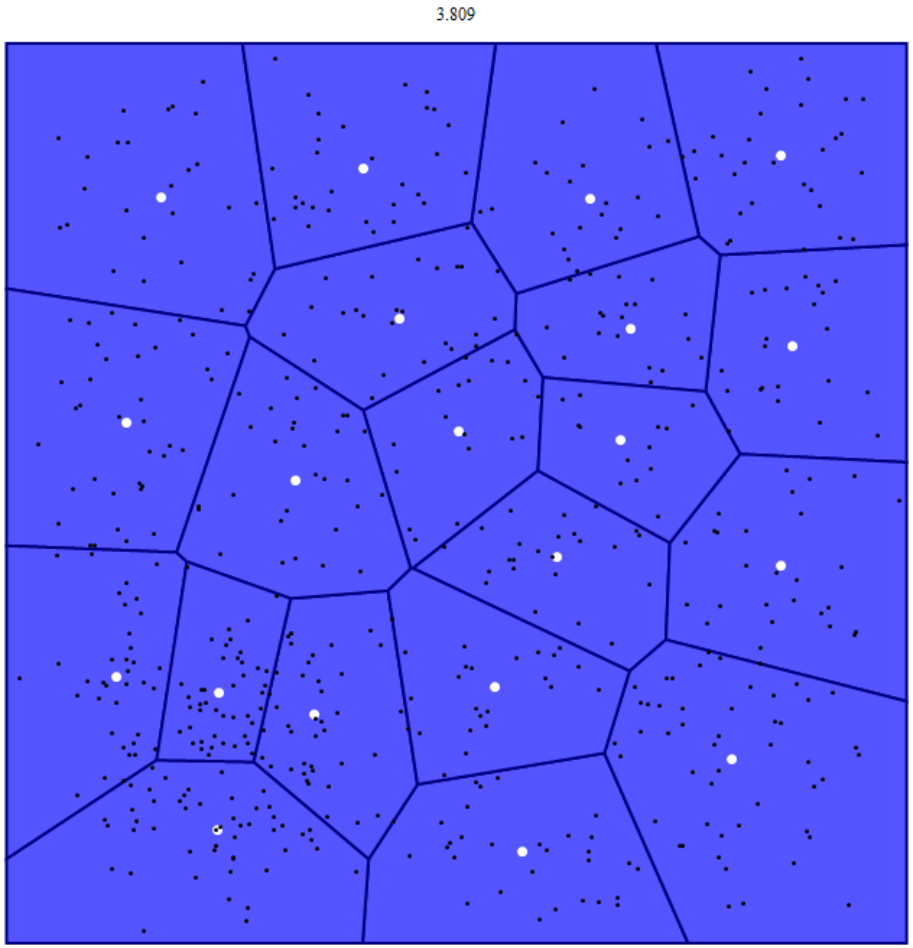

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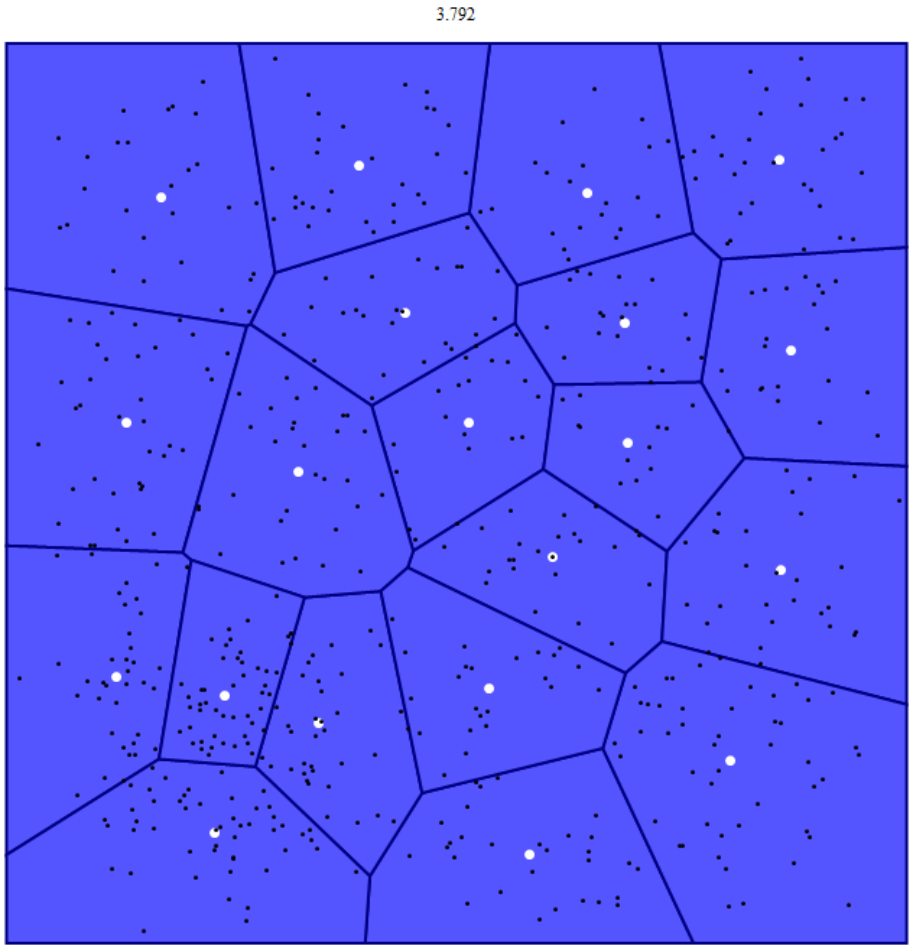

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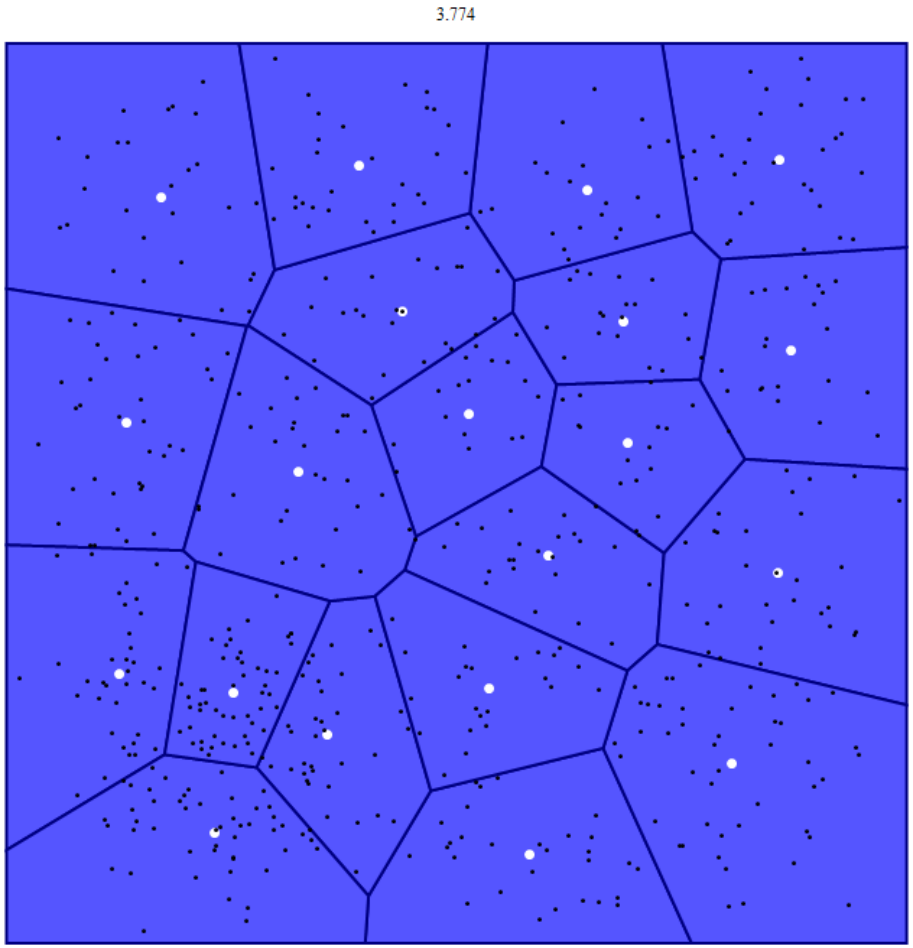

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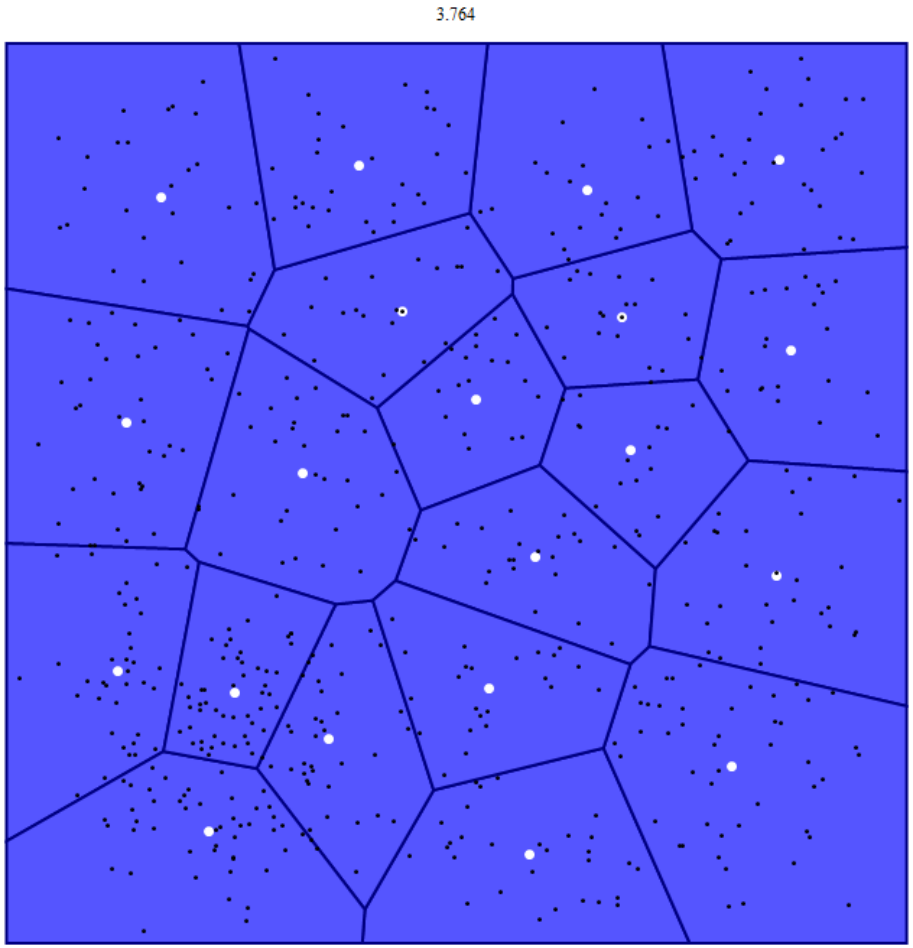

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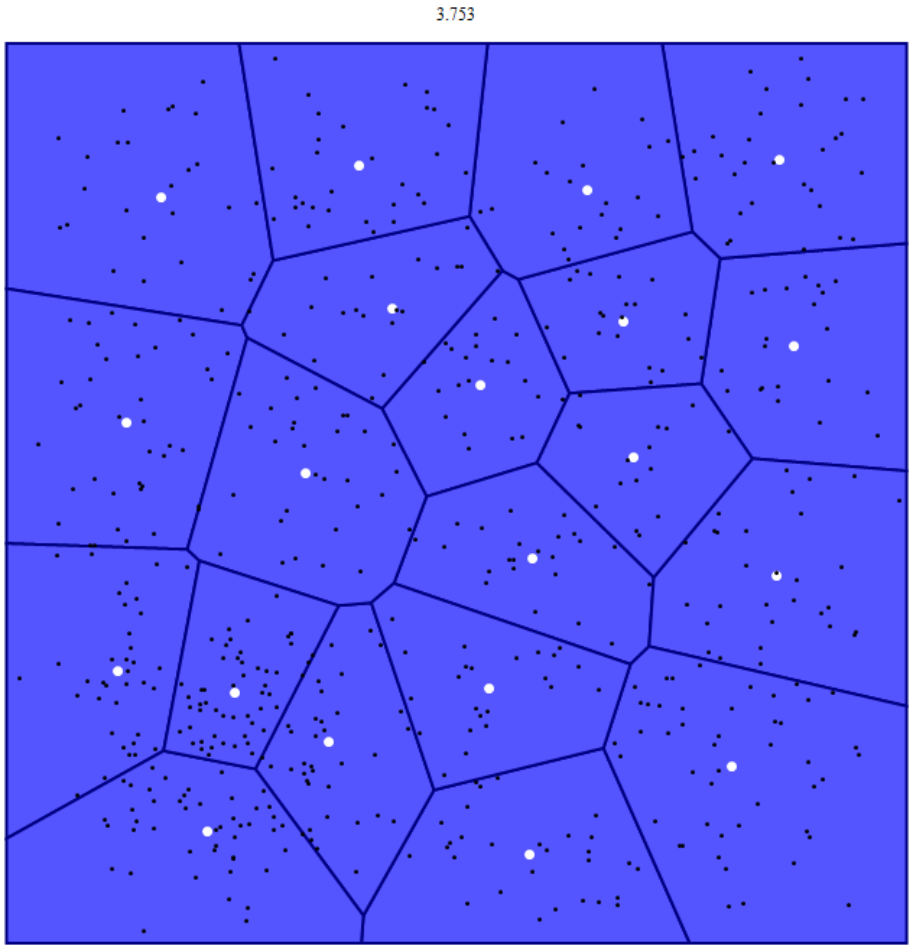

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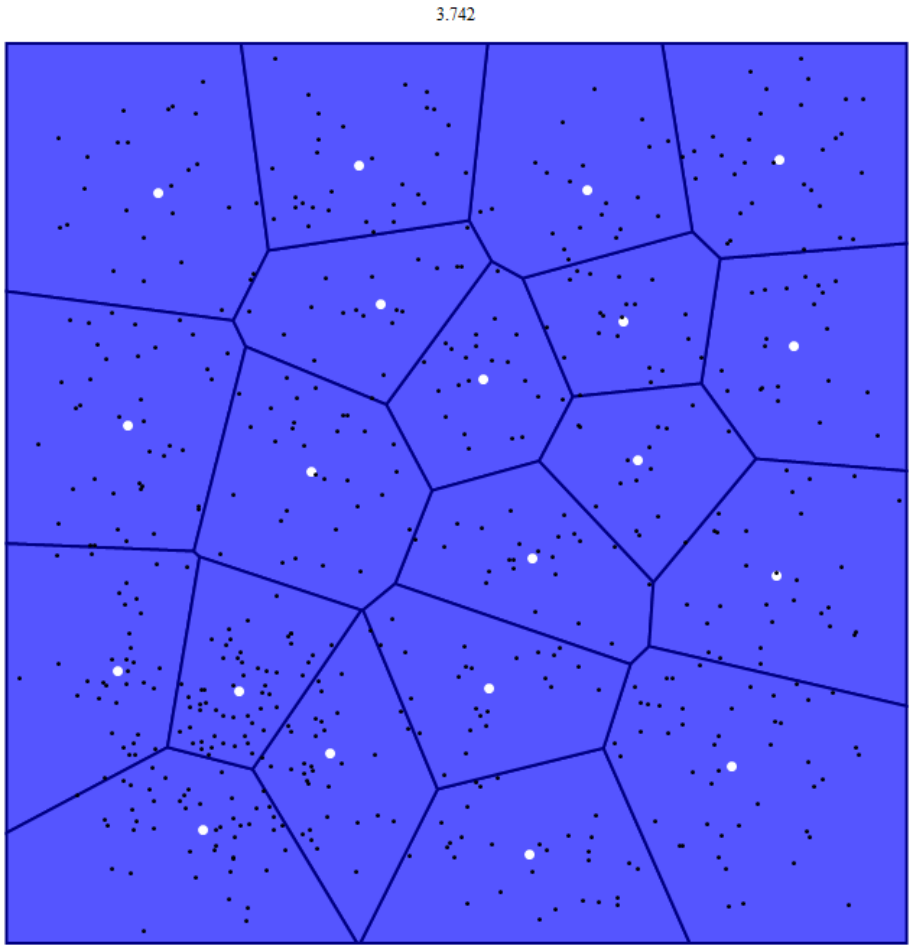

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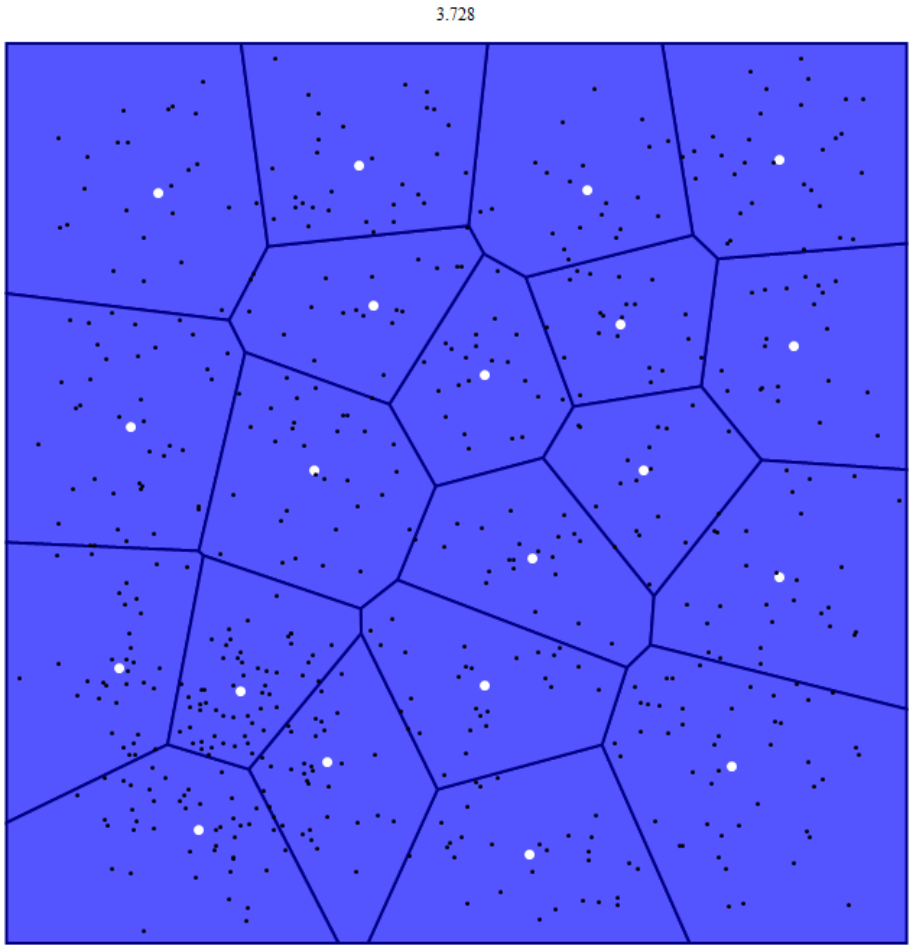

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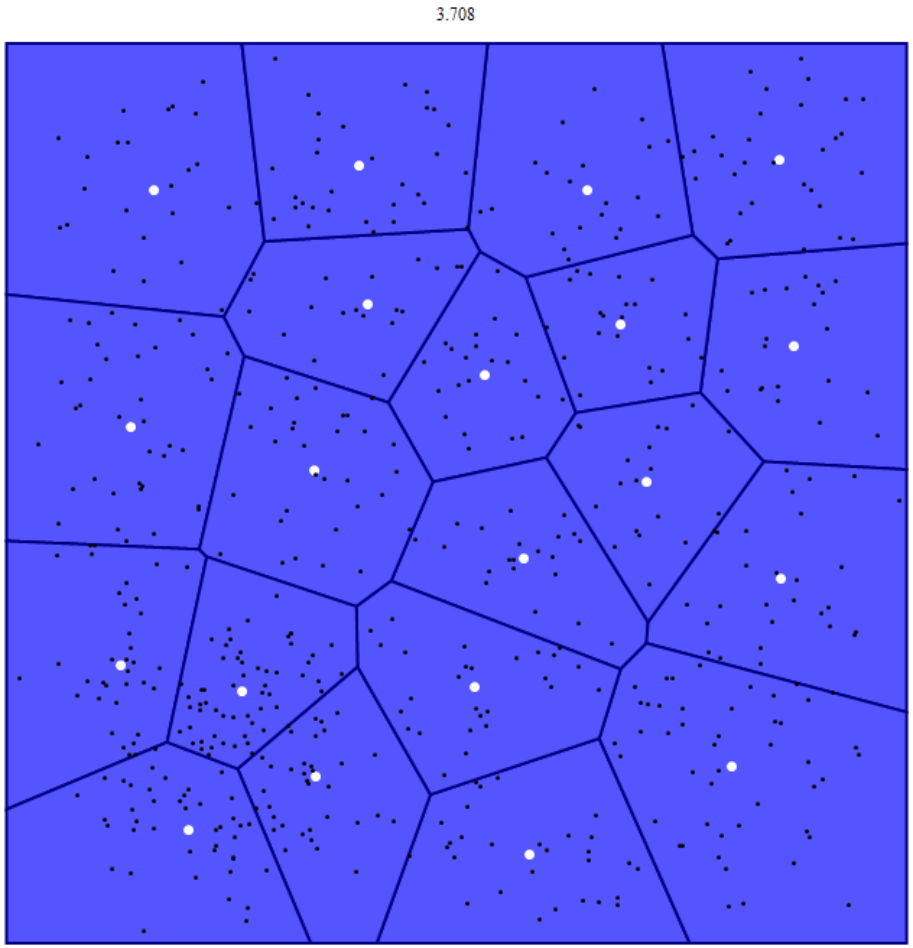

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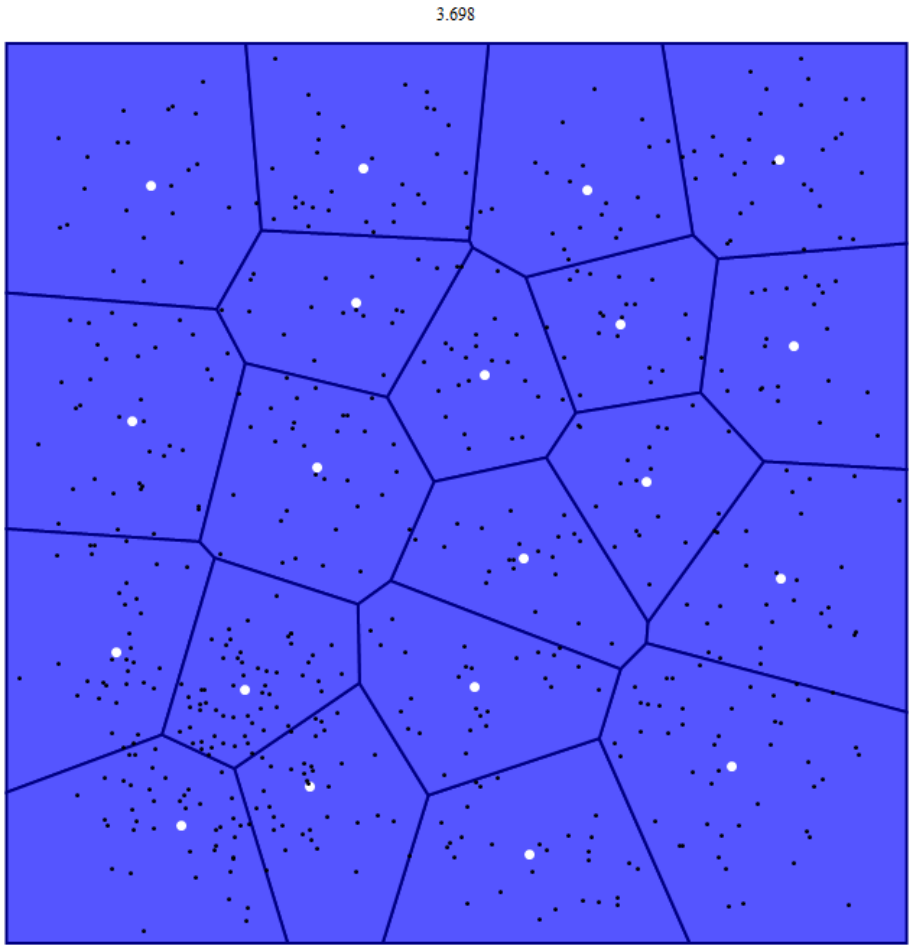

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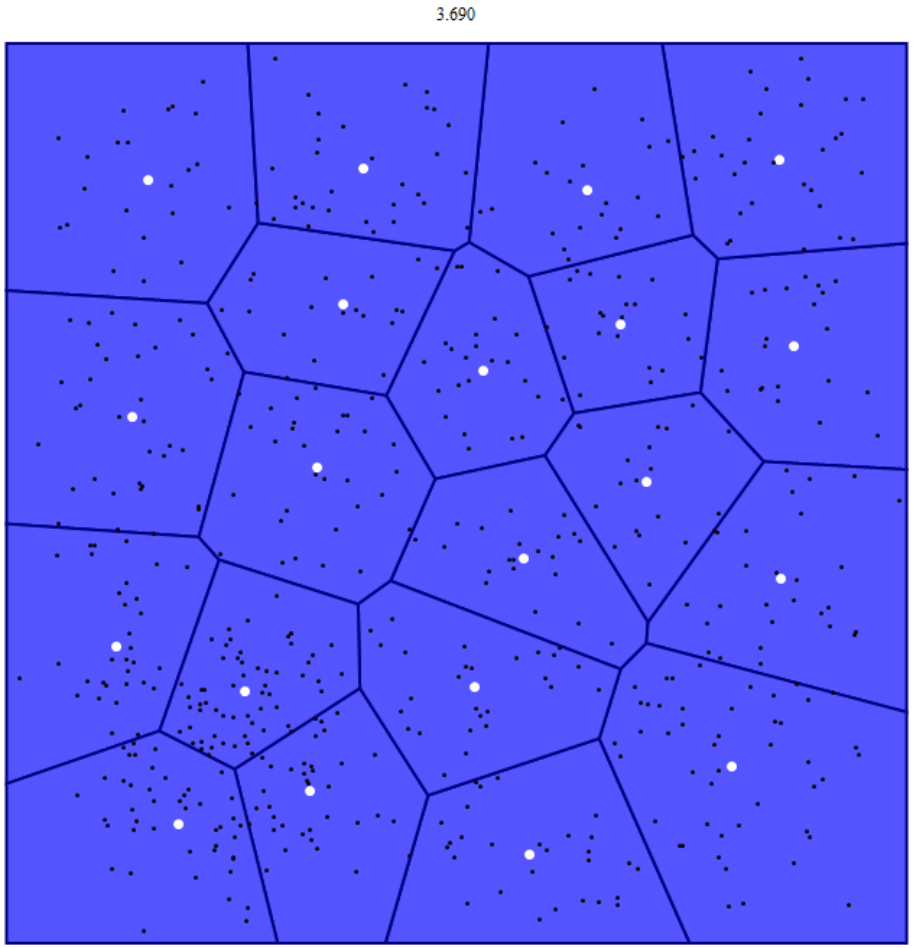

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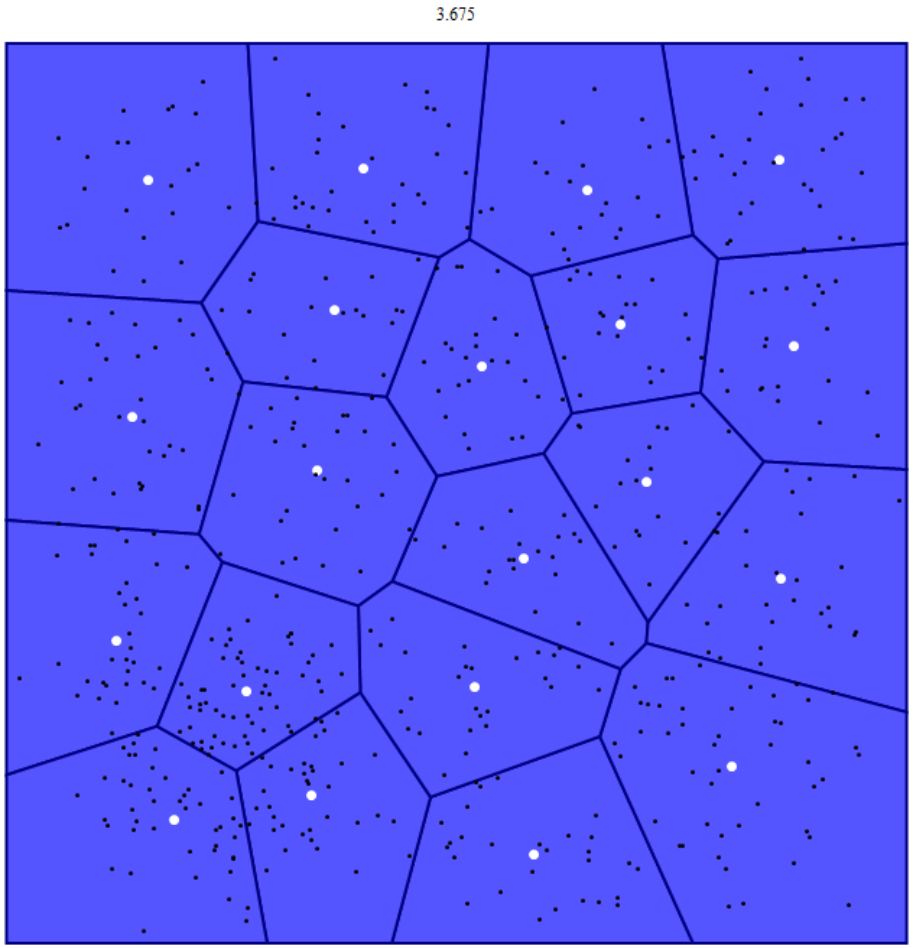

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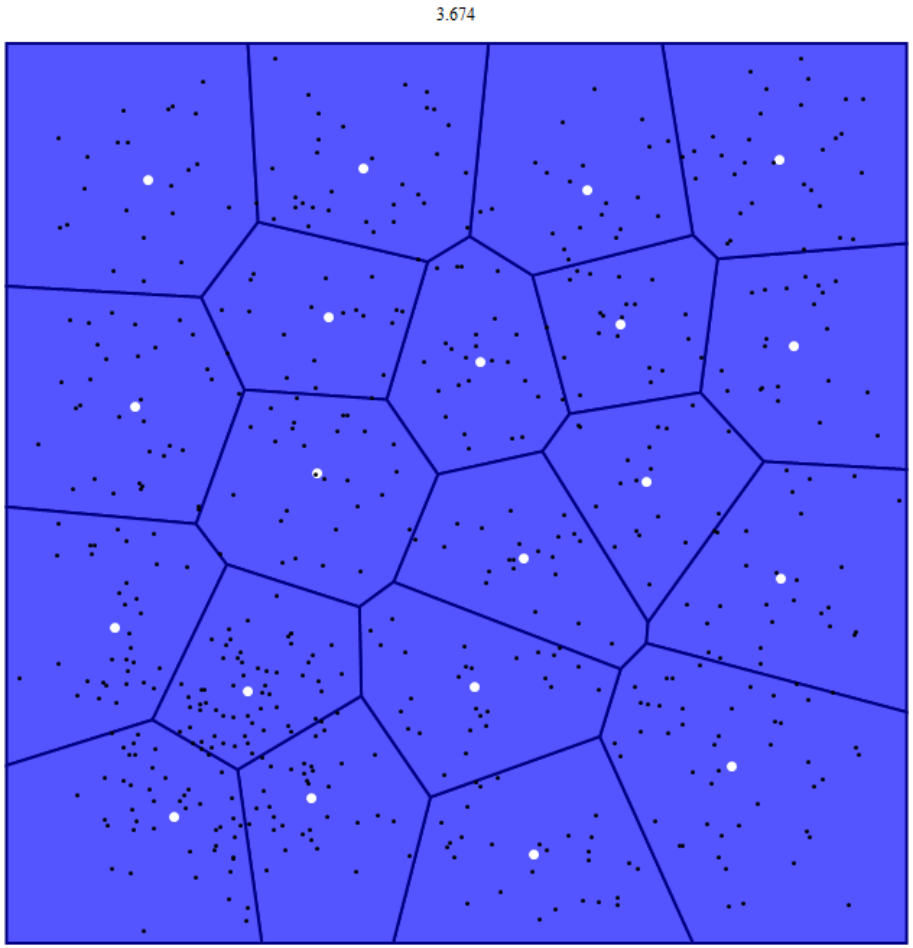

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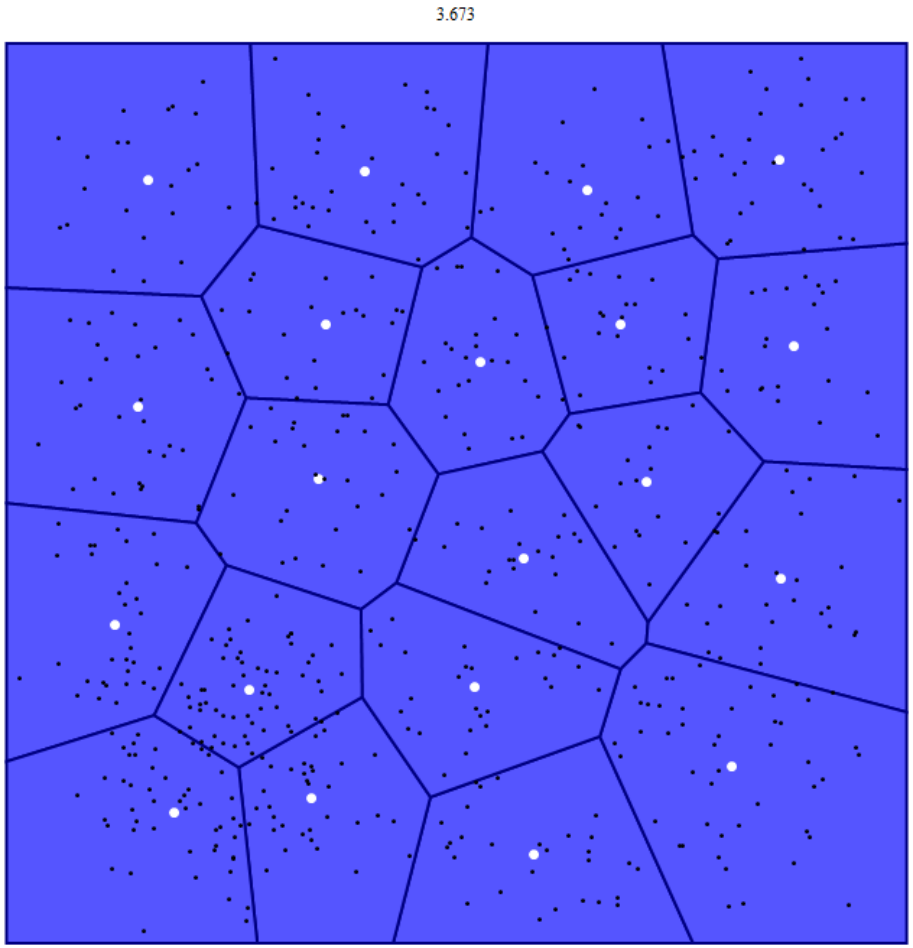

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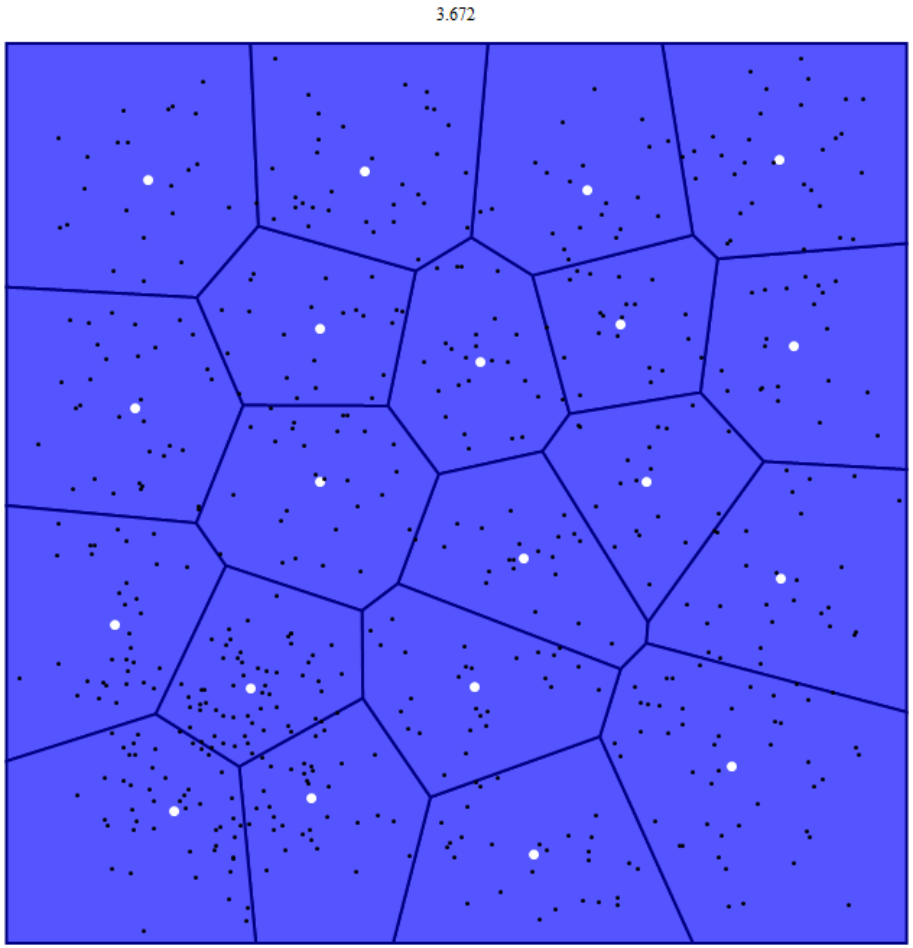

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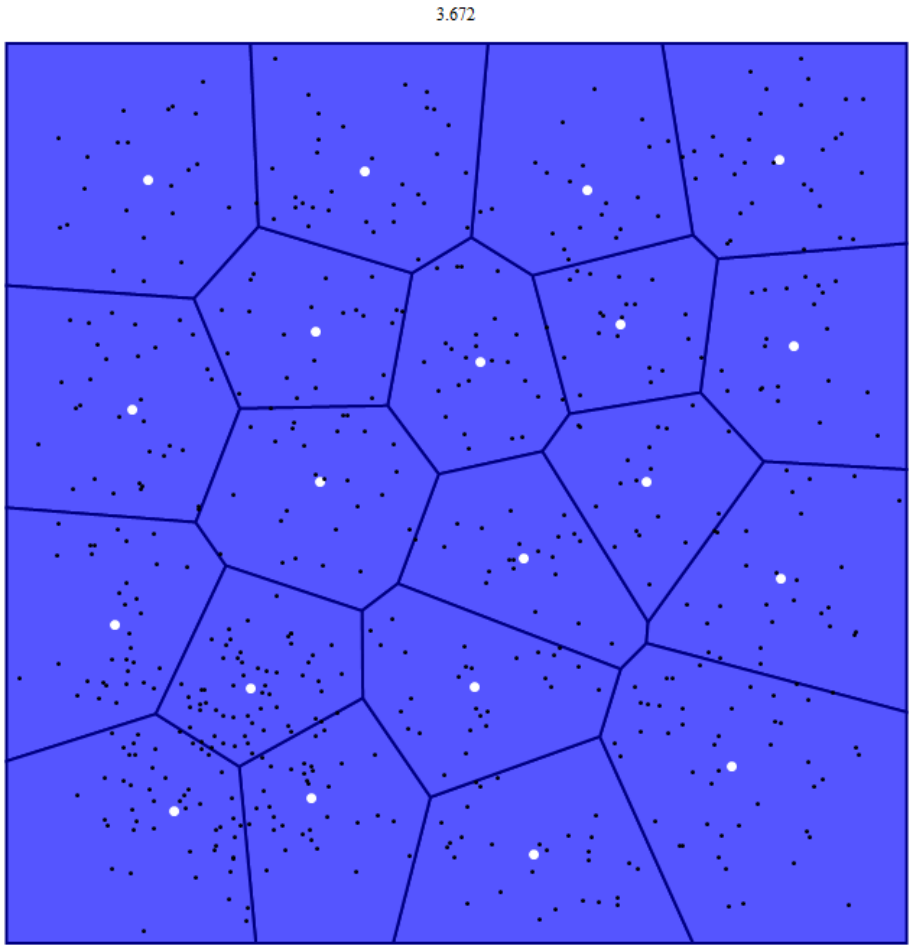

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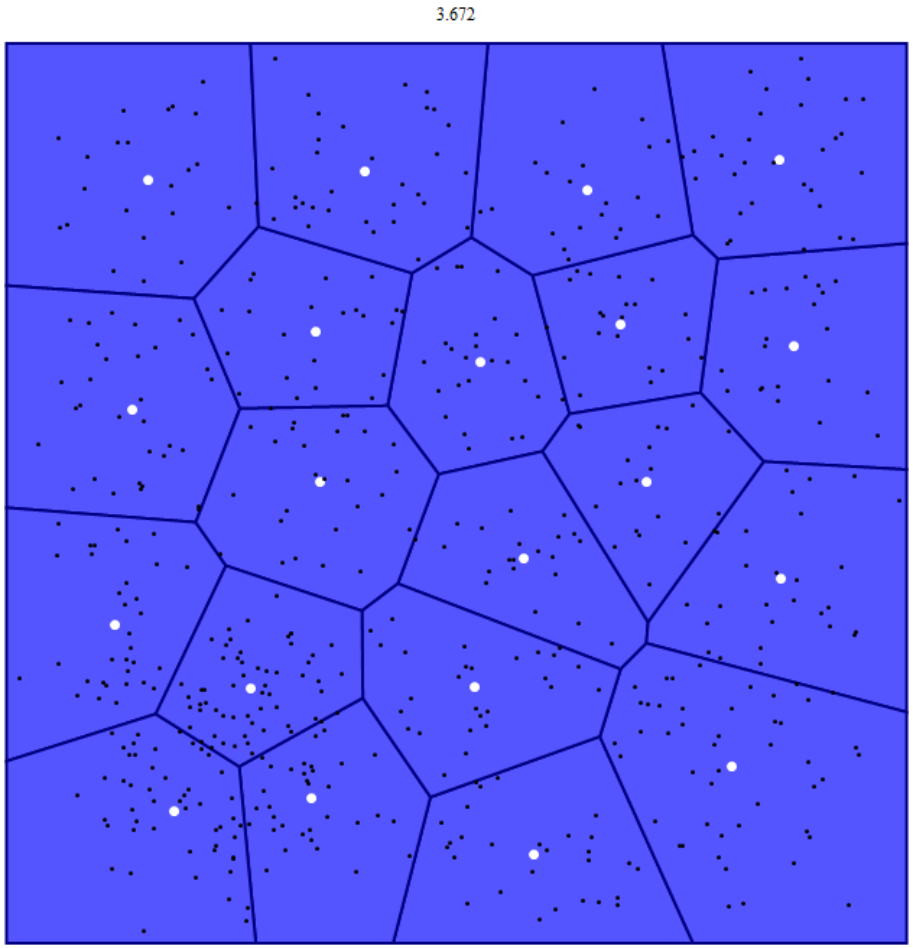

Any Questions?

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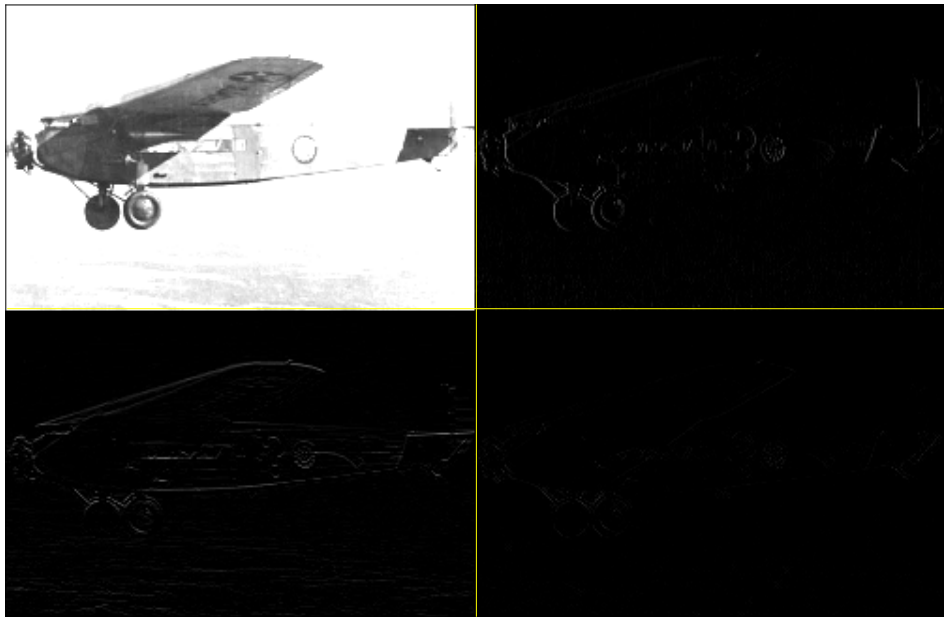

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