A Capstone Course for Future High School Mathematics Teachers

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Overview
Basic Characteristics Important for K–12 Math Teaching
by H. Wu
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- **Precision**: Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.
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- **Definitions:** They are the bedrock of the mathematical structure. They are the platform that supports reasoning. No definitions, no mathematics.
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- **Precision**: Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.

- **Definitions**: They are the bedrock of the mathematical structure. They are the platform that supports reasoning. No definitions, no mathematics.

- **Reasoning**: The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching—and learning—by rote.
Basic Characteristics Important for K–12 Math Teaching by H. Wu
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- **Coherence:** Mathematics is a tapestry in which all the concepts and skills are interwoven. It is all of a piece.
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- **Coherence:** Mathematics is a tapestry in which all the concepts and skills are interwoven. It is all of a piece.

- **Purposefulness:** Mathematics is goal-oriented, and every concept or skill is there for a purpose. Mathematics is not just fun and games.
Common Core State Standards: Mathematical Practices (for Pupils)

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
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Course Structure

Textbooks

Elizabeth G. Bremigan, Ralph J. Bremigan, and John D. Lorch: Mathematics for Secondary School Teachers
Felix Klein:
Elementary Mathematics from an Advanced Standpoint
Arithmetic, Algebra, Analysis
First published in 1908
Course Content:

- From the Natural Numbers to the Real Numbers
- Complex Numbers
- Functions
- Equations and Inequalities

Prerequisite:
Introduction to Logic and Proof
Course Content:

- From the Natural Numbers to the Real Numbers
Course Content:

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Prerequisite:
- Introduction to Logic and Proof
Wu’s Principles and Student Beliefs:
1. Precision

- Students make progress during the course in understanding that they need to transition from being “Math learners” to “Math talkers”.
Student Group Activity I

- Teach a Lesson
Student Group Activity I

- Teach a Lesson
  - Each student group teaches 1–2 sections in the Equations chapter
  - Fellow classmates present homework solutions - lesson presenters critique the solutions.
  - Duration: 50–80 minutes
Student Group Activity I

Teach a Lesson

- Each student group teaches 1–2 sections in the Equations chapter
- Fellow classmates present homework solutions - lesson presenters critique the solutions.
- Duration: 50–80 minutes
- Each group needs to complete a trial run with the instructor before teaching the lesson
Student Group Activity II

- Final Paper & Presentation
Student Group Activity II

- Final Paper & Presentation
  - Countability of algebraic numbers
  - Algebraic numbers form a field
  - Stereographic projection
  - Transcendence of the number $e$ (à la Felix Klein)
  - Cardano-Tartaglia method for solving cubic equations (incl. *Casus Irreducibilis*)
  - Newton’s Method
Wu’s Principles and Student Beliefs:

2. Definitions

- Students do not understand the role of definitions in Mathematics and/or believe that definitions are not important for HS Mathematics.
- Students are not good as “digesting” definitions.
Example: Class Activity

- Define “chair”.
- Define “prime number”.
Example: Class Activity

- Define “chair”.
- Define “prime number”.

Subsequent discussion of the differences between the two definitions. . .
Students say that mathematical definitions are more precise than other definitions; the words “descriptive vs. normative” do not surface.
Recall that a relation on a set $X$ is a subset of the Cartesian product $X \times X$. We define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(p, q) \sim (p', q') \iff p + q' = p' + q.$$ 

Show that $\sim$ defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$. 

Example: Group Activity
Example: Group Activity

- Recall that a *relation on a set* $X$ is a subset of the Cartesian product $X \times X$.

We define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(p, q) \sim (p', q') \iff p + q' = p' + q.$$ 

Show that $\sim$ defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

*No student group notices that $X = \mathbb{N} \times \mathbb{N}$. . .*
Wu’s Principles and Student Beliefs:
3. Reasoning

- Based on their own HS experience(?), students will not expect mathematical reasoning from their pupils.
Wu’s Principles and Student Beliefs:
3. Reasoning

- Based on their own HS experience(?), students will not expect mathematical reasoning from their pupils.
- Students do not understand that reasoning with definitions will be the only way for them to “check” the Mathematics they will be teaching.
Example: Class Discussion
(after discussing and practicing the Division Algorithm)

Instructor: Why do rational numbers have terminating or periodic decimal expansions?
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- Instructor: Why do rational numbers have terminating or periodic decimal expansions?  
- Students: Silence...
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- Instructor: Why do rational numbers have terminating or periodic decimal expansions?
- Students: Silence...
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- Student: Our teacher told us.
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(after discussing and practicing the Division Algorithm)

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- Students: Silence. . .
- Students: Silence. . .
- Student: Our teacher told us.

Last semester was the first time that the instructor prompt “How do you compute the decimal expansion of a rational number?” enabled a couple of students to come up with a line of reasoning. . .
Example: Discussion of a Test Problem  
(after finishing the Equations chapter)

- Solve $\log_2 x - \log_2 (1 + \sqrt{x}) = -1$.  

$\log_2 x - \log_2 (1 + \sqrt{x}) = -1$  

$x - (1 + \sqrt{x}) = 2^{-1}$  

$x - 1 - \sqrt{x} = \frac{1}{2}$  

$x^2 - 2x - 1 = 0$  

$x = 1 + \sqrt{2}$  

$x = 1 - \sqrt{2}$ (Not a valid solution since $\sqrt{x}$ is defined for $x \geq 0$)  

$x = 1 - \sqrt{2}$ is a valid solution.
Example: Discussion of a Test Problem  
(after finishing the Equations chapter)

- Solve $\log_2 x - \log_2(1 + \sqrt{x}) = -1$.
- $\log_2 \frac{x}{1 + \sqrt{x}} = -1$
Example: Discussion of a Test Problem  
(after finishing the Equations chapter)

- Solve \( \log_2 x - \log_2(1 + \sqrt{x}) = -1 \).
- \( \log_2 \frac{x}{1 + \sqrt{x}} = -1 \)
- \( \frac{x}{1 + \sqrt{x}} = 1 \)
- \( \frac{1}{2} \)
Example: Discussion of a Test Problem
(after finishing the Equations chapter)

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- $\log_2 \frac{x}{1 + \sqrt{x}} = -1$
- $\frac{x}{1 + \sqrt{x}} = \frac{1}{2}$
- $2x - 1 = \sqrt{x}$
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- \( 2x - 1 = \sqrt{x} \)
- \( 4x^2 - 4x + 1 = x \)
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(after finishing the Equations chapter)

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- $\log_2 \frac{x}{1 + \sqrt{x}} = -1$
- $\frac{x}{1 + \sqrt{x}} = \frac{1}{2}$
- $2x - 1 = \sqrt{x}$
- $4x^2 - 4x + 1 = x$
- $(4x - 1)(x - 1) = 0$
Example: Discussion of a Test Problem
(after finishing the Equations chapter)

- Solve $\log_2 x - \log_2 (1 + \sqrt{x}) = -1$.
- $\log_2 \frac{x}{1 + \sqrt{x}} = -1$
- $\frac{x}{1 + \sqrt{x}} = 2$
- $2x - 1 = \sqrt{x}$
- $4x^2 - 4x + 1 = x$
- $(4x - 1)(x - 1) = 0$

*Why is $x = 1$ a solution, while $x = 1/4$ isn’t?*
Example: Discussion of a Test Problem
(after finishing the Equations chapter)

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Example: Discussion of a Test Problem
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  Domain of the equation: $\mathbb{R}^+$
Example: Discussion of a Test Problem
(after finishing the Equations chapter)

- Solve \( \log_2 x - \log_2(1 + \sqrt{x}) = -1 \).
  
  Domain of the equation: \( \mathbb{R}^+ \)

- \( \iff \log_2 \frac{x}{1 + \sqrt{x}} = -1 \)

- \( \iff \frac{x}{1 + \sqrt{x}} = \frac{1}{2} \)

- \( \iff 2x - 1 = \sqrt{x} \)
Example: Discussion of a Test Problem
(after finishing the Equations chapter)

- Solve $\log_2 x - \log_2(1 + \sqrt{x}) = -1$.
  
  Domain of the equation: $\mathbb{R}^+$

- $\iff \log_2 \frac{x}{1 + \sqrt{x}} = -1$

- $\iff \frac{x}{1 + \sqrt{x}} = \frac{1}{2}$

- $\iff \frac{x}{1 + \sqrt{x}} = \frac{1}{2}$

- $\iff 2x = \sqrt{x}$

- $\implies 4x^2 - 4x + 1 = x$

- $\iff (4x - 1)(x - 1) = 0$

*Why is $x = 1$ a solution, while $x = 1/4$ isn’t?*
Wu’s Principles and Student Beliefs:
4. Coherence

- Students do not seem to have been exposed much to the idea that “Mathematics is a tapestry in which all the concepts and skills are interwoven”.
- Mathematics seems to have been taught to them as a collection of more or less unrelated mathematical subjects.
Example: Algebra

- Being essentially an Algebra course, the course probably contributes to the problem.
- The connection between “Modern Algebra” and “HS Algebra” is stressed.
Wu’s Principles and Student Beliefs:
5. Purposefulness

- Intrinsic motivation seems to be a completely new concept to students.
- Their HS and college experience for the most part seem to have been mathematically “purposeless”.
Example: Questions for Discussion

- What are real numbers good for? Where/when in the HS curriculum do we need real numbers?
- What were complex numbers invented for?
- Why are surjectivity and injectivity important properties of functions?
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Example: Questions for Discussion

- What are real numbers good for? Where/when in the HS curriculum do we need real numbers?
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- Why are surjectivity and injectivity important properties of functions?
Example: Questions During the Trial Run
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- What is the purpose of the material you present?
Example: Questions During the Trial Run

- What is the purpose of the material you present?
- Does the material you present tell a story?
Example: Questions During the Trial Run

- What is the purpose of the material you present?
- Does the material you present tell a story?
- Why do you present this example? What purpose does it serve?
References


2 Hung-Hsi Wu, *The Mathematics K–12 Teachers Need to Know*. 
http://math.berkeley.edu/~wu/Schoolmathematics1.pdf
