A Plaidoyer for Boolean Algebra

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I propose that the basics of Boolean Algebra become a part of the standard *Introduction to Proof* course.

Benefits

- “Sets” and “Logic”, the two classical examples in Boolean Algebra, are **main topics** in an *Introduction to Proof* course.

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- “Sets” and “Logic”, the two classical examples in Boolean Algebra, are **main topics** in an *Introduction to Proof* course.
- Boolean Algebra provides a manageable and complete example of an **axiomatic system.**
- Boolean Algebra is **abstract.**
- Boolean Algebra levels the playing field - students usually have had **no prior exposure** to Boolean Algebra.

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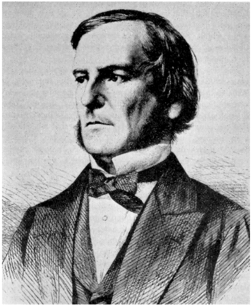
- I do not lecture on Boolean Algebra; instead the problems are part of the **written homework** assigned throughout the semester.



George Boole
(1815–1864)

Edward V. Huntington
(1874–1952)

The Axiomatic System

A *Boolean Algebra* is a set $\mathcal{B}$ together with two “connectives” $\sqcap$ and $\sqcup$ satisfying the following properties:

- **Closure Laws**:
  1. If A and B are two elements in $\mathcal{B}$, then $A \sqcap B$ is also an element in $\mathcal{B}$.
  2. If A and B are two elements in $\mathcal{B}$, then $A \sqcup B$ is also an element in $\mathcal{B}$.

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- **Commutative Laws**:
  1. $A \sqcap B = B \sqcap A$ for all elements A and B in $\mathcal{B}$.
  2. $A \sqcup B = B \sqcup A$ for all elements A and B in $\mathcal{B}$.

The Axiomatic System II

- **Distributive Laws**:
  1. $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C)$ for all elements A, B and C in $\mathcal{B}$.
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- Note: The Associative Laws can be deduced from the other five Boolean Algebra Laws.

The Axiomatic System III

- **Identity Laws**:

  There are elements $N \in \mathcal{B}$ (called the *null element*) and $O \in \mathcal{B}$ (the *one element*) such that

  1. $A \sqcap N = N$ and $A \sqcap O = A$ for all elements A in $\mathcal{B}$.
  2. $A \sqcup O = O$ and $A \sqcup N = A$ for all elements A in $\mathcal{B}$.

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- **Complement Law**:

  For every element A in $\mathcal{B}$ there is an element B in $\mathcal{B}$ such that $A \sqcap B = N$ and $A \sqcup B = O$.

The two classical examples:

1. **Observation:** Let X be an arbitrary set. Then its power set $\mathcal{P}(X)$ with the connectives $\cap$ (in the role of $\sqcap$) and $\cup$ (in the role of $\sqcup$) forms a Boolean Algebra.

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2. **Problem:** Show that

$$\mathcal{S}_1 = \{P \wedge \neg P;\ P,\ \neg P;\ P \vee \neg P\}$$

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3. **Problem:** Find the Boolean Algebra $\mathcal{S}_2$ generated by two free statements P and Q. How many elements does $\mathcal{S}_2$ have?

A third elementary example:

For a natural number n, let $\mathcal{D}_n$ denote the set of the divisors of n. For example, $\mathcal{D}_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$ and $\mathcal{D}_{12} = \{1, 2, 3, 4, 6, 12\}$. For $m, n \in \mathbb{N}$ let $m \sqcap n$ denote the greatest common divisor of n and m, and $m \sqcup n$ their least common multiple. For instance $6 \sqcap 4 = 2$ and $6 \sqcup 4 = 12$. It turns out that $\mathcal{D}_{42}$ with these two operations $\sqcap$ and $\sqcup$ forms a Boolean Algebra, while $\mathcal{D}_{12}$ does **not**.

4. **Problem:** Verify the Boolean Algebra Laws for $\mathcal{D}_{42}$.
5. **Problem:** Show that $\mathcal{D}_{12}$ does not form a Boolean Algebra.
6. **Problem:** Conjecture for which values of n the set $\mathcal{D}_n$ forms a Boolean Algebra.

The topic of Boolean Algebra can be revisited when the course "covers" partial orders:

7. **Problem:** Consider the relation "$\preceq$" on a Boolean Algebra $\mathcal{B}$ defined by

$$A \preceq B \quad \Leftrightarrow \quad A \sqcup B = B$$

for $A, B \in \mathcal{B}$. Prove that $\preceq$ is reflexive, anti-symmetric and transitive.

8. **Problem:** Consider the Boolean Algebra $\mathcal{S}_1$. Draw a *Hasse diagram* for $\mathcal{S}_1$ endowed with the partial order $\preceq$.

The crucial definition needed to lead to the representation theorem for finite Boolean Algebras is the following:

> *Let $\mathcal{B}$ be a Boolean Algebra with null-element N, partially ordered by $\preceq$. We say that $A \in \mathcal{B}$ is an* ATOM *of $\mathcal{B}$ if N is an immediate predecessor of A.*

9. **Problem:** Find all atoms of $\mathcal{P}(\{1, 2, 3, 4\})$.
10. **Problem:** Find all atoms of $\mathcal{D}_{42}$.
11. **Problem:** Find a Boolean Algebra with 8 elements that is a subset of $\mathcal{P}(\{1, 2, 3, 4\})$, but **not** the power set of a three-element subset of $\{1, 2, 3, 4\}$, then find its atoms and draw its Hasse diagram.

A sequence of four more problems studying atoms in a Boolean Algebra is needed before students are ready for the “big theorem” at the end of the semester.

Marshall H. Stone (1903–1989)

Let $\mathcal{B}$ be a finite Boolean Algebra with k atoms for some $k \in \mathbb{N}$, and let $\mathcal{A}$ denote the power set of the set of all atoms of $\mathcal{B}$.
Let

$$\alpha(B) = \{A \in \mathcal{B} \mid A \preceq B \text{ and } A \text{ is an atom of } \mathcal{B}\}.$$

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12. **Problem:** Show that the function $\alpha : \mathcal{B} \to \mathcal{A}$ is a bijection.
13. **Problem:** $\mathcal{B}$ has 2^k elements.
14. **Problem:** Show that the identities $\alpha(B \sqcup B') = \alpha(B) \cup \alpha(B')$ and $\alpha(B \sqcap B') = \alpha(B) \cap \alpha(B')$ hold for all $B, B' \in \mathcal{B}$.
15. **Problem:** Additionally, $\alpha(N) = \emptyset$ and $\alpha(O)$ is the set of all atoms of $\mathcal{B}$.

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- R.L. Goodstein, *Boolean Algebra*. Dover Pub., 2007.
- Edward V. Huntington, *Sets of Independent Postulates for the Algebra of Logic*. Transactions of the American Mathematical Society 5 (1904), pp. 288-309.
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- Marshall H. Stone, *The Theory of Representation for Boolean Algebras*. Transactions of the American Mathematical Society 40 (1936), pp. 37-111.
- J.E. Whitesitt, Boolean Algebra and Its Applications. Dover Pub., 1995

The complete *Boolean Algebra theorem sequence* is available at `helmut.knaust.info/presentations/BA.pdf`