

A Plaidoyer for Boolean Algebra

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I propose that the basics of Boolean Algebra become a part of the standard *Introduction to Proof* course.

Benefits

- “Sets” and “Logic”, the two classical examples in Boolean Algebra, are **main topics** in an *Introduction to Proof* course.

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- Boolean Algebra provides a manageable and complete example of an **axiomatic system**.
- Boolean Algebra is **abstract**.

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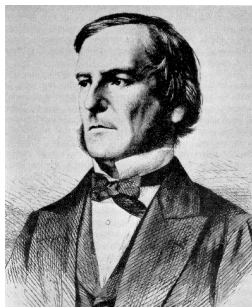
- “Sets” and “Logic”, the two classical examples in Boolean Algebra, are **main topics** in an *Introduction to Proof* course.
- Boolean Algebra provides a manageable and complete example of an **axiomatic system**.
- Boolean Algebra is **abstract**.
- Boolean Algebra levels the playing field - students usually have had **no prior exposure** to Boolean Algebra.

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- I use a “**Moore-style theorem sequence**”, leading to the finite version of *Marshall Stone’s* Representation Theorem for Boolean Algebras.
- I do not lecture on Boolean Algebra; instead the problems are part of the **written homework** assigned throughout the semester.



George Boole
(1815–1864)



Edward V. Huntington
(1874–1952)

The Axiomatic System

A *Boolean Algebra* is a set \mathcal{B} together with two “connectives” \sqcap and \sqcup satisfying the following properties:

- **Closure Laws:**

- 1 If A and B are two elements in \mathcal{B} , then $A \sqcap B$ is also an element in \mathcal{B} .
- 2 If A and B are two elements in \mathcal{B} , then $A \sqcup B$ is also an element in \mathcal{B} .

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- **Commutative Laws:**

- 1 $A \sqcap B = B \sqcap A$ for all elements A and B in \mathcal{B} .
- 2 $A \sqcup B = B \sqcup A$ for all elements A and B in \mathcal{B} .

The Axiomatic System II

- **Distributive Laws:**

① $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all elements A , B and C in \mathcal{B} .

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- **Note:** The Associative Laws can be deduced from the other five Boolean Algebra Laws.

The Axiomatic System III

- **Identity Laws:**

There are elements $N \in \mathcal{B}$ (called the *null element*) and $O \in \mathcal{B}$ (the *one element*) such that

- 1 $A \sqcap N = N$ and $A \sqcap O = A$ for all elements A in \mathcal{B} .
- 2 $A \sqcup O = O$ and $A \sqcup N = A$ for all elements A in \mathcal{B} .

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- **Complement Law:**

For every element A in \mathcal{B} there is an element B in \mathcal{B} such that $A \sqcap B = N$ and $A \sqcup B = O$.

The two classical examples:

- 1 **Observation:** Let X be an arbitrary set. Then its power set $\mathcal{P}(X)$ with the connectives \cap (in the role of \sqcap) and \cup (in the role of \sqcup) forms a Boolean Algebra.

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- 2 **Problem:** Show that

$$\mathcal{S}_1 = \{P \wedge \neg P; P, \neg P; P \vee \neg P\}$$

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③ **Problem:** Find the Boolean Algebra \mathcal{S}_2 generated by two free statements P and Q . How many elements does \mathcal{S}_2 have?

A third elementary example:

For a natural number n , let \mathcal{D}_n denote the set of the divisors of n . For example, $\mathcal{D}_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$ and $\mathcal{D}_{12} = \{1, 2, 3, 4, 6, 12\}$. For $m, n \in \mathbb{N}$ let $m \sqcap n$ denote the greatest common divisor of n and m , and $m \sqcup n$ their least common multiple. For instance $6 \sqcap 4 = 2$ and $6 \sqcup 4 = 12$. It turns out that \mathcal{D}_{42} with these two operations \sqcap and \sqcup forms a Boolean Algebra, while \mathcal{D}_{12} does **not**.

- 4 **Problem:** Verify the Boolean Algebra Laws for \mathcal{D}_{42} .
- 5 **Problem:** Show that \mathcal{D}_{12} does not form a Boolean Algebra.
- 6 **Problem:** Conjecture for which values of n the set \mathcal{D}_n forms a Boolean Algebra.

The topic of Boolean Algebra can be revisited when the course “covers” partial orders:

- 7 **Problem:** Consider the relation “ \preceq ” on a Boolean Algebra \mathcal{B} defined by

$$A \preceq B \iff A \sqcup B = B$$

for $A, B \in \mathcal{B}$. Prove that \preceq is reflexive, anti-symmetric and transitive.

- 8 **Problem:** Consider the Boolean Algebra \mathcal{S}_1 . Draw a *Hasse diagram* for \mathcal{S}_1 endowed with the partial order \preceq .

The crucial definition needed to lead to the representation theorem for finite Boolean Algebras is the following:

*Let \mathcal{B} be a Boolean Algebra with null-element N , partially ordered by \preceq . We say that $A \in \mathcal{B}$ is an **ATOM** of \mathcal{B} if N is an immediate predecessor of A .*

- 9 **Problem:** Find all atoms of $\mathcal{P}(\{1, 2, 3, 4\})$.
- 10 **Problem:** Find all atoms of \mathcal{D}_{42} .
- 11 **Problem:** Find a Boolean Algebra with 8 elements that is a subset of $\mathcal{P}(\{1, 2, 3, 4\})$, but **not** the power set of a three-element subset of $\{1, 2, 3, 4\}$, then find its atoms and draw its Hasse diagram.

A sequence of four more problems studying atoms in a Boolean Algebra is needed before students are ready for the “big theorem” at the end of the semester.



Marshall H. Stone
(1903–1989)

Let \mathcal{B} be a finite Boolean Algebra with k atoms for some $k \in \mathbb{N}$, and let \mathcal{A} denote the power set of the set of all atoms of \mathcal{B} .

Let

$$\alpha(\mathcal{B}) = \{A \in \mathcal{B} \mid A \preceq B \text{ and } A \text{ is an atom of } \mathcal{B}\}.$$

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- 12 **Problem:** Show that the function $\alpha : \mathcal{B} \rightarrow \mathcal{A}$ is a bijection.
- 13 **Problem:** \mathcal{B} has 2^k elements.
- 14 **Problem:** Show that the identities $\alpha(\mathcal{B} \sqcup \mathcal{B}') = \alpha(\mathcal{B}) \cup \alpha(\mathcal{B}')$ and $\alpha(\mathcal{B} \sqcap \mathcal{B}') = \alpha(\mathcal{B}) \cap \alpha(\mathcal{B}')$ hold for all $\mathcal{B}, \mathcal{B}' \in \mathcal{B}$.
- 15 **Problem:** Additionally, $\alpha(\mathcal{N}) = \emptyset$ and $\alpha(\mathcal{O})$ is the set of all atoms of \mathcal{B} .

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The complete *Boolean Algebra theorem sequence* is available
at helmut.knaust.info/presentations/BA.pdf