How to Make Teachers out of Mathematics Students

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The International Sun Conference on Teaching and Learning

I don't know.

I don’t know. But I try. . .

Overview

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Fundamental Mathematics from an Advanced Standpoint

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- The vast majority of students will be student-teaching within 6–12 months.

Course Contents

- From the counting numbers to the real numbers
- Complex numbers
- Functions
- Algebra (= Solving Equations)

Fundamental vs. Advanced Standpoint
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  - Cardinality

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- Many students were "bulimic learners" in high school: Whatever they memorized, they have long forgotten.
- Students have experienced mathematics mostly as consumers.
- The compartmentalized structure of learning in college does not encourage students to see connections.

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1. **Precision:** Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.
2. **Definitions:** They are the bedrock of the mathematical structure. They are the platform that supports reasoning. No definitions, no mathematics.
3. **Reasoning:** The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching—and learning—by rote.

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4. **Coherence:** Mathematics is a tapestry in which all the concepts and skills are interwoven. It is all of a piece.
5. **Purposefulness:** Mathematics is goal-oriented, and every concept or skill is there for a purpose. Mathematics is not just fun and games.

Common Core State Standards: Mathematical Practices (for Pupils)

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning

Wu's Principles and Student Beliefs: 1. Precision

- Students make progress during the course in understanding that they need to transition from being “Math learners” to “Math talkers”.

Student Group Activity I

- Teach a Lesson

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  - Each group needs to complete a trial run with the instructor before teaching the lesson

Student Group Activity II

- Final Paper & Presentation

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  - Countability of algebraic numbers
  - Algebraic numbers form a field
  - Stereographic projection
  - Transcendence of the number e (à la Felix Klein)
  - Cardano-Tartaglia method for solving cubic equations (incl. *Casus Irreducibilis*)

Wu's Principles and Student Beliefs:
2. Definitions

- Students do not understand the role of definitions in Mathematics and/or believe that definitions are not important for HS Mathematics.
- Students are not good as "digesting" definitions.

Example: Group Activity

- Recall that a *relation on a set X* is a subset of the Cartesian product $X \times X$.

  We define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ as follows:

  $$(p, q) \sim (p', q') \Leftrightarrow p + q' = p' + q.$$

  Show that $\sim$ defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

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Basically no student group notices that $X = \mathbb{N} \times \mathbb{N}\ldots$

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- Based on their own HS experience(?), students will not expect mathematical reasoning from their pupils.
- Students do not understand that reasoning with definitions will be the only way for them to "check" the Mathematics they will be teaching.

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(after discussing and practicing the Division Algorithm)

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Last time was the first time that the instructor prompt "How do you compute the decimal expansion of a rational number?" enabled a couple of students to come up with a line of reasoning...

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- $\Leftrightarrow \dfrac{x}{1+\sqrt{x}} = \dfrac{1}{2}$
- $\Leftrightarrow 2x - 1 = \sqrt{x}$
- $\Rightarrow 4x^2 - 4x + 1 = x$
- $\Leftrightarrow (4x-1)(x-1) = 0$

Why is $x = 1$ a solution, while $x = 1/4$ isn't?

Wu's Principles and Student Beliefs: 4. Coherence

- Students do not seem to have been exposed much to the idea that "Mathematics is a tapestry in which all the concepts and skills are interwoven".
- Mathematics seems to have been taught to them as a collection of more or less unrelated mathematical subjects.

Example: Algebra

- Being essentially an Algebra course, the course may contribute to the problem.
- The connection between "Modern Algebra" and "HS Algebra" is stressed.

Wu's Principles and Student Beliefs: 5. Purposefulness

- Intrinsic motivation seems to be a completely new concept to students.
- Their HS and college experience for the most part seem to have been mathematically "purposeless".

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- What were complex numbers invented for?
- Why are surjectivity and injectivity important properties of functions?

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- What is the purpose of the material you present?
- Does the material you present tell a story?
- Why do you present this example? What purpose does it serve?

References

1. Matthew Winsor, *One Model of a Capstone Course for Preservice High School Mathematics Teachers*. Primus 19 (2009), pp. 510–518.
2. Hung-Hsi Wu, *The Mathematics K–12 Teachers Need to Know*. `http://math.berkeley.edu/~wu/Schoolmathematics1.pdf`
3. Hung-Hsi Wu, *The Mis-Education of Mathematics Teachers*. Notices of the AMS 58 (2011).