Multi-Resolution Analysis for the Haar Wavelet: A Minimalist Approach

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I am regularly teaching a discrete wavelets course as part of my department's *Master of Arts in Teaching Mathematics* program.

- Diverse student population:
  - Current high school (and middle school) teachers
  - Recent graduates of our B.S. program,
  - Career switchers
- This is the only required applied mathematics course in the program.

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2. Learn how to use a computer algebra system for mathematical investigations, as a computational and visualization aid, and for the implementation of mathematical algorithms
3. Develop an understanding of the theoretical underpinnings of wavelet transforms and their applications

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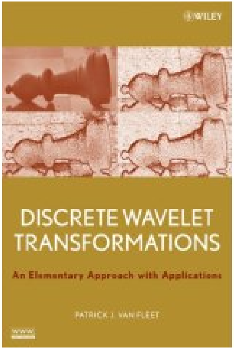
- Open the mathematical toolbox to find solutions (Fourier analysis)
- Find better tools (wavelets) or improve the existing ones.
- Understand what else the tools can do in the given context (e.g., edge detection).

Student Prerequisites:

- A thorough understanding of Calculus
- Some familiarity with matrices
- Mathematical maturity
- Willingness to learn *Mathematica*

Textbook:

Patrick Van Fleet, *Discrete Wavelet Transformations: An Elementary Approach with Applications*. Wiley-Interscience.



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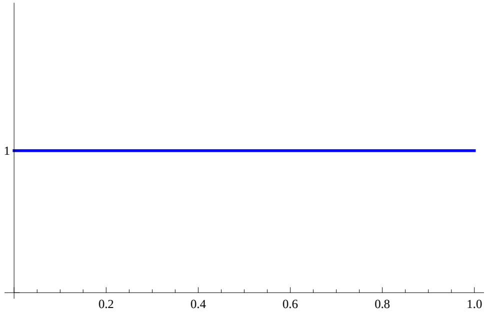
- Easier than the general case (unit interval, only one direction)
- Reinforces the notion of orthogonality of functions
- Seamlessly leads to discrete transformations via matrices

$$\phi(x) = \begin{cases} 1, & \text{if } x \in [0,1) \\ 0, & \text{if } x < 0 \text{ or } x \geq 1 \end{cases}$$



$\phi : \mathbb{R} \to \mathbb{R}$ is called the HAAR SCALING FUNCTION. Throughout we will identify $\phi(x)$ with its restriction to $[0, 1)$.

Notice that we obtain the dilation equation

$$\phi(x) = \phi(2x) + \phi(2x - 1).$$

Use this to define a sequence of vector spaces $V_0 \subseteq V_1 \subseteq V_2, \ldots$, generated by the dilations of ϕ, i.e., V_j is spanned by the orthogonal functions

$$2^{j/2}\phi(2^j x), 2^{j/2}\phi(2^j x - 1), \ldots, 2^{j/2}\phi(2^j x - (2^j - 1)).$$

The vector space V_j consists of all functions on $[0, 1)$ that are constant on the standard intervals of length 2^{-j}.

Gram-Schmidt orthogonalization to replace the basis in V_1 by the basis vector in V_0 plus an orthonormal vector $\psi(x)$ in V_1 yields the Haar wavelet function

$$\begin{aligned}\psi(x) &= \phi(2x) - \phi(2x-1) \\ &= \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}) \\ -1, & \text{if } x \in [\frac{1}{2}, 1) \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

Then use dilations of $\psi(x)$ to create the multi-resolution analysis

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}$$

Consider the example $V_4 = V_3 \oplus W_3$.
(V_4 has dimension 16, while both V_3 and W_3 have dimension 8.)
Given a function in V_4, an elementary calculation yields that the coefficient c_k of the corresponding function in W_3 is given by

$$c_k = \frac{a_{2k} - a_{2k+1}}{\sqrt{2}},$$

where a_k denotes the kth coefficient of the function in V_4.
In other words, we obtain the vector representing the function in W_3 by multiplying the vector $(a_0, a_2, \ldots, a_{15})$ of the function in V_4 by the following matrix.

$$\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix}$$

Similarly the coefficients in V_3 are generated by multiplying the coefficients in V_4 by the matrix

$$
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}
$$

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Handout:

helmut.knaust.info/paper/MRA.pdf