Constructing the Real Numbers IBL-Style

Helmut Knaust

Department of Mathematical Sciences
The University of Texas at El Paso

hknaust@utep.edu

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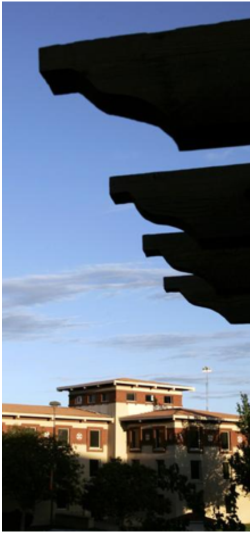
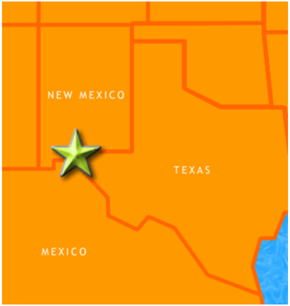
UTEP and Its Students

- $\sim$25,000 students total
- Close to 3,000 graduate students
- 83% Hispanic
- UTEP has Carnegie R1 status



The Closed Loop

- The overwhelming majority of UTEP's students graduate from local public schools.
- A large majority of local teachers are UTEP alumni.



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- Participants take 18 graduate hours of Mathematics courses — designed for MATM students.

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- An "exemplary" applied mathematics course on Discrete Wavelet Transforms
- A "second" Analysis course using Inquiry-based Learning

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- Topic: The axiomatic development of the real number system
- Method: A Moore-style theorem sequence

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- An example of working with an axiomatic system

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- The only course in the graduate program to use IBL formally
- As a model: Give teachers a chance to explore. If they like it, they will let their students explore...

The axiomatic system

Richard Dedekind started by giving the following definition of the set of **Natural Numbers**.

Axioms for $\mathbb{N}$

The natural numbers are a set $\mathbb{N}$ containing a special element called 0, and a function $S : \mathbb{N} \to \mathbb{N}$ satisfying the following axioms:

The function S is called the successor function.

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- D1 S is injective.
- D2 $S(\mathbb{N}) = \mathbb{N} \setminus \{0\}$.
- D3 If a subset M of $\mathbb{N}$ contains 0 and satisfies $S(M) \subseteq M$, then $M = \mathbb{N}$.

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First steps

After letting students prove the **Principle of Induction**, **addition** is defined recursively:

Addition on $\mathbb{N}$

For a fixed but arbitrary $m \in \mathbb{N}$ we define

$$\begin{aligned} m + 0 &:= m \\ m + S(n) &:= S(m+n) \qquad \text{for all } n \in \mathbb{N} \end{aligned}$$

The standard **arithmetic properties** of the natural numbers are established.

One target theorem or this section is the **totality** of the order:

The Order on $\mathbb{N}$ is total

For all $m, n \in \mathbb{N},\ m \leq n$ or $n \leq m$.

The **general recursion principle** is placed at the end of the section on natural numbers:

Recursion Principle

Let A be an arbitrary set, and let $a \in A$ and a function $f : A \to A$ be given. Then there exists a unique map $\varphi : \mathbb{N} \to A$ satisfying

1. $\varphi(0) = a$, and
2. $\varphi \circ S = f \circ \varphi$.

The students then use the recursion principle to prove, among other results, the essential **uniqueness of the natural numbers**:

Uniqueness of $\mathbb{N}$

Suppose that $\mathbb{N}$, $S : \mathbb{N} \to \mathbb{N}$ and 0 satisfy Axioms (D1)–(D3), and that $\mathbb{N}'$, $S' : \mathbb{N}' \to \mathbb{N}'$ and $0'$ satisfy Axioms (D1)–(D3) as well.

Then there is a bijection $\varphi : \mathbb{N} \to \mathbb{N}'$ such that

1. $\varphi(0) = 0'$, and
2. $\varphi \circ S = S' \circ \varphi$.

The course spends quite some time on the development of the **integers** $\mathbb{Z}$ as well, but does not spend much time on the set of **rational numbers** $\mathbb{Q}$.

The crucial definition for the construction of the real numbers is **Dedekind cut**:

Dedekind Cut

A partition (L, U) of $\mathbb{Q}$ is called a *Dedekind cut*, if the following properties hold:

1. If $a \in L$ and $b \in U$, then $a < b$.
2. U has no minimal element.

The real numbers $\mathbb{R}$ is then defined as the set of all Dedekind cuts.

While establishing the arithmetic properties of Dedekind cuts is unpleasant (and only partially established by the students), defining the order on Dedekind cuts is **simple**,

Order on Dedekind cuts

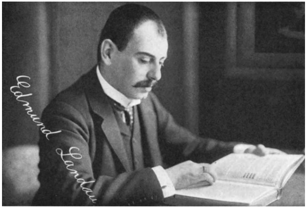
$$(L_1, U_1) \leq (L_2, U_2) \quad \text{if} \quad L_1 \subseteq L_2.$$

and the target theorem of this section has a "**beautiful**" proof:

Completeness of $\mathbb{R}$

Show that $\mathbb{R}$, the set of all Dedekind cuts, satisfies the following property: All subsets of $\mathbb{R}$ that are bounded from above have a least upper bound.

The Heroes



Edmund Landau
(1877-1938)

Richard Dedekind
(1831-1916)

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- All teachers are improving substantially throughout the semester.
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- The material takes up 2/3 to 3/4 of a standard long semester.

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- Every student could only start two problems per week.
- This is a viable alternative to teaching this course in person.

References:

- Stephen Abbott: Understanding Analysis. Springer, 2nd edition (2015)
- Heinz-Dieter Ebbinghaus, et al.: Zahlen, Springer (1983)
- Edward D. Gaughan: Introduction to Analysis, AMS, 5th edition (2009)
- Edmund Landau: Grundlagen der Analysis. Chelsea, 4th edition (1965)

The complete *Theorem sequence*[1] is available from `hknaust@utep.edu`, or directly at `helmut.knaust.info/presentations/AnalysisII.pdf`.