Wu's Principles: Teaching a Capstone Course for Future High School Teachers

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AMATYC
50 YEARS OF IDEAS

Overview

Wu

Hung-Hsi Wu

Professor Emeritus
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- The vast majority of students will be student-teaching within a year.

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- In the past, we offered the course every semester...

Course Contents

- From the counting numbers to the real numbers
- Complex numbers
- Functions
- Algebra (= Solving Equations)

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  - Cardinality

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- Many students were "bulimic learners" in high school: Whatever they memorized, they have long forgotten.
- Students have experienced mathematics mostly as consumers, not as producers.
- The compartmentalized structure of learning in college does not encourage students to see connections.

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1. **Precision:** Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.
2. **Definitions:** They are the bedrock of the mathematical structure. They are the platform that supports reasoning. No definitions, no mathematics.
3. **Reasoning:** The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching—and learning—by rote.

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4. **Purposefulness:** Mathematics is goal-oriented, and every concept or skill is there for a purpose.
5. **Coherence:** Mathematics is a tapestry in which all the concepts and skills are interwoven. It is all of a piece.

Wu's Principles and Student Beliefs:
1. Precision

- Students make progress during the course in understanding that they need to transition from being "Math learners" to "Math talkers".

Student Group Activity I

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  - Duration: 80 minutes
  - Each group needs to complete a trial run with the instructor before teaching the lesson

Student Group Activity II

- Final Presentation & Paper - Some Topics:

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- Final Presentation & Paper - Some Topics:
  - Countability of algebraic numbers
  - Algebraic numbers form a field
  - Stereographic projection
  - Cardano-Tartaglia method for solving cubic equations (incl. *Casus Irreducibilis*)
  - An exact formula for $\sin 1^\circ$

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- Recall that a *relation on a set* X is a subset of the Cartesian product $X \times X$.

  We define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ as follows:

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  Show that $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

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No student group notices that $X = \mathbb{N} \times \mathbb{N}$...

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- Based on their own HS experience(?), students will not expect mathematical reasoning from their pupils.
- Students need to understand that reasoning with definitions will be the only way for them to "check" the Mathematics they will be teaching.

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Why is $x = 1$ a solution, while $x = 1/4$ isn't?

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- $\Leftrightarrow (4x-1)(x-1) = 0 \quad \wedge \quad 2x - 1 \geq 0$
- $\Leftrightarrow x = 1$

Wu's Principles and Student Beliefs:
4. Purposefulness

- Intrinsic motivation seems to be a completely new concept to students.
- Their HS and college experience for the most part seem to have been mathematically "purposeless".

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- What is the purpose of the material you present?
- Does the material you present tell a story?
- Why do you present this example? What purpose does it serve?

Wu's Principles and Student Beliefs:
5. Coherence

- Students do not seem to have been exposed much to the idea that "Mathematics is a tapestry in which all the concepts and skills are interwoven".
- Mathematics seems to have been taught to them as a collection of more or less unrelated mathematical subjects.

Topic: Algebra

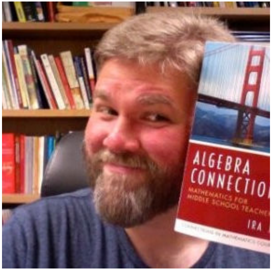
- Being essentially an Algebra course, the course may contribute to the problem.
- The connection between "Modern Algebra" and "HS Algebra" is stressed.

A Link between Complex Numbers and Geometry: Classification of Isometries in the Complex Plane

- **Theorem.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an arbitrary isometry. Then there exist* $z_0 \in \mathbb{C}$ *and* $\theta \in \mathbb{R}$ *such that* $f(z) = z_0 + ze^{i\theta}$ *or* $f(z) = z_0 + \overline{z}e^{i\theta}$.

Credits

Matthew Winsor



Heinz Althoff

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