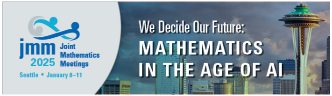
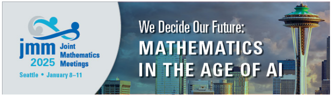
Transformation-based Geometry Through Complex Numbers: Teaching a Capstone Course for Future High School Teachers

Helmut Knaust
Department of Mathematical Sciences
The University of Texas at El Paso

January 11, 2025



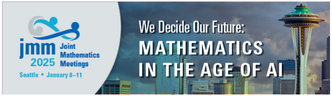
Overview



jmm 2025
Joint Mathematics Meetings
Seattle • January 8–11
We Decide Our Future:
MATHEMATICS
IN THE AGE OF AI

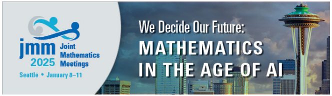
UTEP and Its Students

- 25,000+ students total
- $\sim$ 3,700 graduate students



UTEP and Its Students

- 25,000+ students total
- $\sim$ 3,700 graduate students
- 84% Hispanic

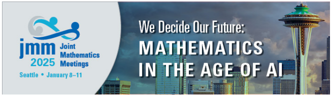


UTEP and Its Students

- 25,000+ students total
- $\sim$ 3,700 graduate students
- 84% Hispanic
- UTEP has Carnegie-R1 status

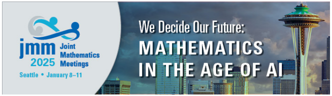
Fundamental Mathematics from an Advanced Standpoint

- The only course in our department taken by future secondary Mathematics teachers that specifically addresses High School Mathematics.



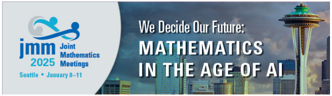
Fundamental Mathematics from an Advanced Standpoint

- The only course in our department taken by future secondary Mathematics teachers that specifically addresses High School Mathematics.
- The prerequisite is an *Introduction to Proof* class. Some students may have taken a number theory and/or geometry class.



Fundamental Mathematics from an Advanced Standpoint

- The only course in our department taken by future secondary Mathematics teachers that specifically addresses High School Mathematics.
- The prerequisite is an *Introduction to Proof* class. Some students may have taken a number theory and/or geometry class.
- The course is almost exclusively taken by juniors and seniors.

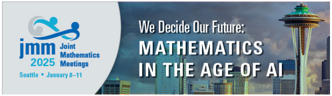


Fundamental Mathematics from an Advanced Standpoint

- The only course in our department taken by future secondary Mathematics teachers that specifically addresses High School Mathematics.
- The prerequisite is an *Introduction to Proof* class. Some students may have taken a number theory and/or geometry class.
- The course is almost exclusively taken by juniors and seniors.
- The vast majority of students will be student-teaching within a year.

Course Contents

- From the counting numbers to the real numbers
- **Complex numbers**
- Functions
- Algebra (= Solving Equations)



Fundamental vs. Advanced Standpoint
Example: Complex Numbers

- In what context have students seen complex numbers?

Fundamental vs. Advanced Standpoint
Example: Complex Numbers

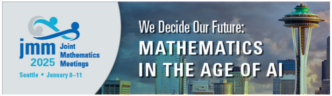
- In what context have students seen complex numbers?
  - To solve quadratic equations

Fundamental vs. Advanced Standpoint
Example: Complex Numbers

- In what context have students seen complex numbers?
    - To solve quadratic equations
    - To solve certain systems of linear differential equations

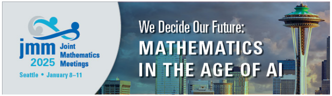
Fundamental vs. Advanced Standpoint
Example: Complex Numbers

- In what context have students seen complex numbers?
    - To solve quadratic equations
    - To solve certain systems of linear differential equations
- Advanced standpoint:



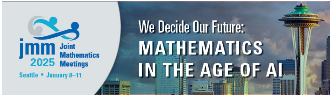
Fundamental vs. Advanced Standpoint
Example: Complex Numbers

- In what context have students seen complex numbers?
  - To solve quadratic equations
  - To solve certain systems of linear differential equations
- Advanced standpoint:
  - $\mathbb{C}$ is a field of numbers just like $\mathbb{R}$, albeit without a nice order



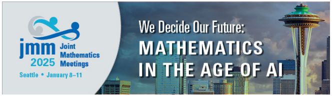
Fundamental vs. Advanced Standpoint
Example: Complex Numbers

- In what context have students seen complex numbers?
  - To solve quadratic equations
  - To solve certain systems of linear differential equations
- Advanced standpoint:
  - $\mathbb{C}$ is a field of numbers just like $\mathbb{R}$, albeit without a nice order
  - $\mathbb{C}$ is algebraically closed: The Fundamental Theorem of Algebra

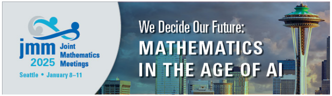


Fundamental vs. Advanced Standpoint
Example: Complex Numbers

- In what context have students seen complex numbers?
  - To solve quadratic equations
  - To solve certain systems of linear differential equations
- Advanced standpoint:
  - $\mathbb{C}$ is a field of numbers just like $\mathbb{R}$, albeit without a nice order
  - $\mathbb{C}$ is algebraically closed: The Fundamental Theorem of Algebra
  - The complex plane

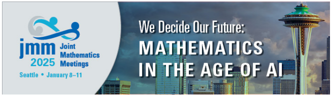


Basic Characteristics
of a “Working” Mathematics Classroom
by H. Wu



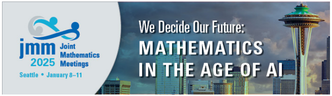
Basic Characteristics of a "Working" Mathematics Classroom by H. Wu

1. **Precision:** Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.



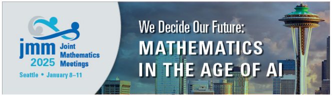
Basic Characteristics of a “Working” Mathematics Classroom by H. Wu

1. **Precision:** Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.
2. **Definitions:** They are the bedrock of the mathematical structure. They are the platform that supports reasoning. No definitions, no mathematics.



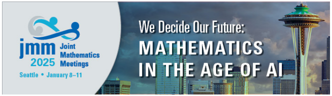
Basic Characteristics of a Mathematics Classroom by H. Wu II

4. **Reasoning:** The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching—and learning—by rote.



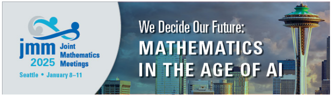
Basic Characteristics of a Mathematics Classroom by H. Wu II

4. **Reasoning:** The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching—and learning—by rote.
5. **Purposefulness:** Mathematics is goal-oriented, and every concept or skill is there for a purpose.



Basic Characteristics of a Mathematics Classroom by H. Wu II

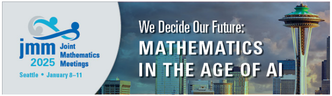
4. **Reasoning:** The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching—and learning—by rote.
5. **Purposefulness:** Mathematics is goal-oriented, and every concept or skill is there for a purpose.
6. **Coherence:** Mathematics is a tapestry in which all the concepts and skills are interwoven. It is all of a piece.



An Example of “Coherence” –
The First Baby Step of the Erlangen Program

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is called an *isometry* if it satisfies

$$|f(w) - f(z)| = |w - z| \text{ for all } w, z \in \mathbb{C}.$$

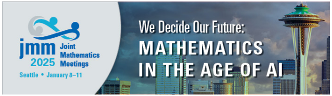


An Example of “Coherence” –
The First Baby Step of the Erlangen Program

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is called an *isometry* if it satisfies

$$|f(w) - f(z)| = |w - z| \text{ for all } w, z \in \mathbb{C}.$$

Examples. ???

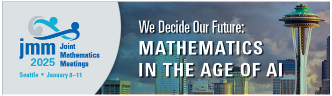


An Example of "Coherence" –
The First Baby Step of the Erlangen Program

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is called an *isometry* if it satisfies

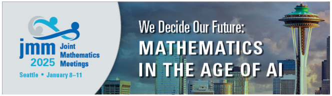
$$|f(w) - f(z)| = |w - z| \text{ for all } w, z \in \mathbb{C}.$$

Examples. Translations, ???, Rotations, Reflections.



Erlangen Program II

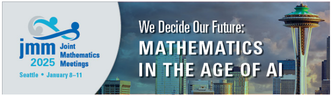
Theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary isometry. Then there exist $z_0 \in \mathbb{C}$ and $\theta \in \mathbb{R}$ such that $f(z) = z_0 + ze^{i\theta}$ or $f(z) = z_0 + \overline{z}e^{i\theta}$.



Erlangen Program II

Theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary isometry. Then there exist $z_0 \in \mathbb{C}$ and $\theta \in \mathbb{R}$ such that $f(z) = z_0 + ze^{i\theta}$ or $f(z) = z_0 + \overline{z}e^{i\theta}$.

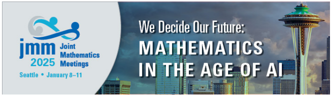
Starting with an arbitrary isometry f, the isometry $g(z) = f(z) - f(0)$ has 0 as a fixed point.



Erlangen Program II

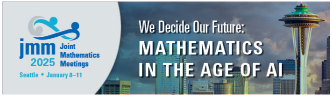
Theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary isometry. Then there exist $z_0 \in \mathbb{C}$ and $\theta \in \mathbb{R}$ such that $f(z) = z_0 + ze^{i\theta}$ or $f(z) = z_0 + \overline{z}e^{i\theta}$.

Starting with an arbitrary isometry f, the isometry $g(z) = f(z) - f(0)$ has 0 as a fixed point. It follows from the isometry condition that then $|g(1)| = 1$, and so for a suitable rotation around the origin, the isometry $h(z) = g(z)e^{-i\theta}$ satisfies $h(0) = 0$ and $h(1) = 1$.



Erlangen Program III

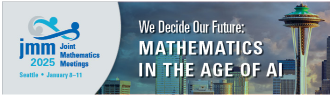
Lemma. Let $h : \mathbb{C} \to \mathbb{C}$ be an isometry that satisfies $h(0) = 0$ and $h(1) = 1$. Then $h(z) = z$ or $h(z) = \overline{z}$.



Erlangen Program III

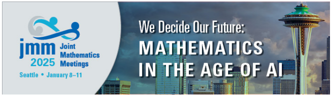
Lemma. Let $h : \mathbb{C} \to \mathbb{C}$ be an isometry that satisfies $h(0) = 0$ and $h(1) = 1$. Then $h(z) = z$ or $h(z) = \overline{z}$.

Reversing the steps on the previous slide we obtain the desired result.



Multiple Representations of $\mathbb{C}$

- Algebraic representation as numbers of the form $a + bi$
- Complex numbers in polar form: $z = re^{i\theta}$
- The complex plane
- The Riemann sphere via stereographic projection
- The complex numbers as certain real 2×2 matrices

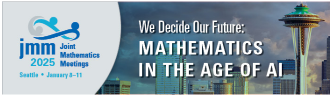


An Example of “Purposefulness” – $\mathbb{C}$ as real 2×2 Matrices

Theorem. The matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

(with $a, b \in \mathbb{R}$) form a field isomorphic to $\mathbb{C}$.



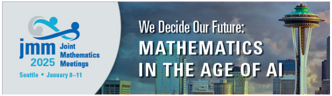
An Example of "Purposefulness" – $\mathbb{C}$ as real 2×2 Matrices

Theorem. The matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

(with $a, b \in \mathbb{R}$) form a field isomorphic to $\mathbb{C}$.

Here 1 *corresponds to the identity matrix and* $\pm i$ *corresponds to* $\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$.



$\mathbb{H}$ as Complex 2×2 Matrices

Theorem. The matrices of the form

$$\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

(with $z, w \in \mathbb{C}$) form a 4-dimensional (over $\mathbb{R}$) division algebra, known as $\mathbb{H}$, the set of **quaternions**.

$\mathbb{H}$ as Complex 2×2 Matrices

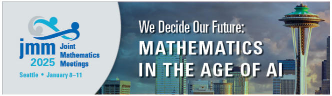
Theorem. The matrices of the form

$$\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

(with $z, w \in \mathbb{C}$) form a 4-dimensional (over $\mathbb{R}$) division algebra, known as $\mathbb{H}$, the set of **quaternions**.

Here the basis elements are $\vec{1}$, $\vec{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$\vec{j} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, *and* $\vec{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.



References

1. Hung-Hsi Wu, *The Mis-Education of Mathematics Teachers*. Notices of the AMS 58 (2011).
2. Matthew Winsor, *One Model of a Capstone Course for Preservice High School Mathematics Teachers*. Primus 19 (2009), pp. 510–518.
3. Tristan Needham, *Visual Complex Analysis*, Clarendon Press, Oxford (1997).
4. Heinz-Dieter Ebbinghaus, et al. *Zahlen*, Springer-Verlag, Berlin (1983).

Contact

Helmut Knaust:

`hknaust@utep.edu`
`http://helmut.knaust.info`